Solutions to Integrals

Mathematical Sciences Society

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1 Integral 3: Justin Stevens

Compute the integral

$$\lim_{x\to\pi}\frac{\int_x^{\pi\cos^2 x}\frac{e^\alpha \alpha}{d}\alpha}{\int_\pi^x\int_\beta^0\gamma\sqrt{\gamma\sqrt[3]{\gamma\sqrt[4]{\cdots}}}d\gamma d\beta}.$$

To begin with, we evaluate the denominator. Observe that

$$\gamma\sqrt{\gamma\sqrt[3]{\gamma\sqrt[4]{\cdots}}}=\gamma^1\gamma^{\frac{1}{2}}\gamma^{\frac{1}{6}}\gamma^{\frac{1}{24}}\cdots\gamma^{\frac{1}{n!}}\cdots.$$

Recall that $e=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots$, thus

$$\gamma\sqrt{\gamma\sqrt[3]{\gamma\sqrt[4]{\cdots}}}=\gamma^{1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\cdots}=\gamma^{e-1}.$$

Therefore $\int_\beta^0\gamma^{e-1}d\gamma=\frac{\gamma^e}{\gamma}\Big|_\beta^0=\frac{-\beta^e}{e}$. We now have an integral of the form

$$\lim_{x\to\pi}\frac{\int_x^{\pi\cos^2 x}\frac{e^\alpha \alpha}{d}\alpha}{\int_\pi^x\int_\beta^0\gamma\sqrt{\gamma\sqrt[3]{\gamma\sqrt[4]{\cdots}}}d\gamma d\beta}=\lim_{x\to\pi}\frac{\int_x^{\pi\cos^2 x}\frac{e^\alpha}{\alpha}d\alpha}{\int_\pi^x\frac{-\beta^e}{e}d\beta}.$$

We now use L'Hospital's rule to see

$$\lim_{x\to\pi}\frac{\int_x^{\pi\cos^2 x}\frac{e^\alpha}{\alpha}d\alpha}{\int_\pi^x\frac{-\beta^e}{e}d\beta}=\frac{\frac{d}{dx}\int_x^{\pi\cos^2 x}\frac{e^\alpha}{\alpha}d\alpha\Big|_{x=\pi}}{\frac{d}{dx}\int_\pi^x\frac{-\beta^e}{e}d\beta\Big|_{x=\pi}}.$$

We now use the Fundamental Theorem of Calculus to see

$$\begin{aligned}\frac{d}{dx}\int_x^{\pi\cos^2 x}\frac{e^\alpha}{\alpha}d\alpha d\alpha\Big|_{x=\pi}&=\left(\frac{d}{dx}\int_0^{\pi\cos^2 x}\frac{e^\alpha}{\alpha}d\alpha-\frac{d}{dx}\int_0^x\frac{e^\alpha}{\alpha}d\alpha\right)\Big|_{x=\pi}\\&=\left(\pi\sin(2x)e^{\pi\cos^2 x}\pi\cos^2 x-\frac{e^x}{x}\right)\Big|_{x=\pi}\\&=-\frac{e^\pi}{\pi}.\end{aligned}$$

Similarly, $\frac{d}{dx}\int_{\pi}^{x}\frac{-\beta^e}{e}d\beta\Big|_{x=\pi} = \frac{-x^e}{e}\Big|_{x=\pi} = \frac{-\pi^e}{e}$. Therefore,

$$\lim_{x\to\pi}\frac{\int_x^{\pi\cos^2 x}\frac{e^\alpha\alpha}{d}\alpha}{\int_\pi^x\int_\beta^0\gamma\sqrt{\gamma\sqrt[3]{\gamma\sqrt[4]{\cdots}}}d\gamma d\beta} = \frac{-\frac{e^\pi}{\pi}}{\frac{-\pi^e}{e}} = \boxed{\frac{e^{\pi+1}}{\pi^{e+1}}}.$$

2 Integral 4: Zhi Han

2.1 Solution 1: Differentiation Under the Integral Sign

The technique of *differentiation under the integral sign* is a powerful technique for solving integrals, as seen in Surely You're Joking:

> One thing I never did learn was contour integration. I had learned to do integrals by various methods shown in a book that my high school physics teacher Mr. Bader had given me. One day he told me to stay after class. "Feynman," he said, "you talk too much and you make too much noise. I know why. You're bored. So I'm going to give you a book. You go up there in the back, in the corner, and study this book, and when you know everything that's in this book, you can talk again." So every physics class, I paid no attention to what was going on with Pascal's Law, or whatever they were doing. I was up in the back with this book: Advanced Calculus, by Woods. Bader knew I had studied Calculus for the Practical Man a little bit, so he gave me the real works—it was for a junior or senior course in college.
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> It had Fourier series, Bessel functions, determinants, elliptic functions—all kinds of wonderful stuff that I didn't know anything about. That book also showed how to *differentiate parameters under the integral sign*—it's a certain operation. It turns out that's not taught very much in the universities; they don't emphasize it.
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> But I caught on how to use that method, and I used that one damn tool again and again. So because I was self-taught using that book, I had peculiar methods of doing integrals. The result was, when guys at MIT or Princeton had trouble doing a certain integral, it was because they couldn't do it with the standard methods they had learned in school. If it was contour integration, they would have found it; if it was a simple series expansion, they would have found it. Then I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me.

We need to evaluate the integral

$$\int_0^1 \frac{t^3 - 1}{\ln t} dt \tag{1}$$

We are given two hints:

Hint 1 (Differentiation under the integral sign), also known as Leibniz Integral Rule:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(x,t)\mathrm{d}t = \int_a^b \frac{\partial}{\partial x} f(x,t)\mathrm{d}t} \tag{2}$$

Hint 2.

$$\frac{d}{dx} t^x = t^x \ln t \tag{3}$$

This integral cannot be solved by elementary tricks. However, the hints imply that we should use the method of *differentiation under the integral sign*, which is Hint 1.

Hint 1 suggests that we should differentiate the integral, so we may define a function

$$g(x) = \int_0^1 \frac{t^x - 1}{\ln t} dt \tag{4}$$

Hint 2 suggests that we should replace t^3 with t^x in the integral to cancel out the annoying $\ln(x)$ on the bottom.

Note that $g(3)$ corresponds to the original integral. Applying differentiation under the integral sign.

$$g'(x) = \int_0^1 \frac{\partial}{\partial x} \frac{t^x - 1}{\ln t} dt = \int_0^1 \frac{t^x \ln t}{\ln t} dt = \left. \frac{t^{x+1}}{x+1} \right|_0^1 = \frac{1}{x+1} \tag{5}$$

It follows that

$$g(x) = \ln|x+1| + C \tag{6}$$

To determine C note that $g(0) = 0$. Now we can solve for $C = 0$. So $g(x) = \ln|x+1|$, so $\boxed{g(3) = \ln(4)}$.

Sources:

https://brilliant.org/wiki/differentiate-through-the-integral/

https://math.stackexchange.com/questions/253910/the-integral-that-stumped-feynman

2.2 Solution 2: Frullani's Theorem

An even more clever solution uses Frullani's theorem.

https://en.wikipedia.org/wiki/Frullani_integral

$$\int_0^1 \frac{t^3 - 1}{\ln t} dt \tag{7}$$

First, we use u-substitution:

$$t = e^{-y} \quad dt = -e^{-y} dy \tag{8}$$

Then the integral becomes

$$\int_{\infty}^{0} -e^{-y} dy \frac{e^{-3y} - 1}{y} \tag{9}$$

$$\int_{0}^{\infty} \frac{e^{-4y} - e^{-y}}{y} dy \tag{10}$$

Applying Frullani's theorem,

$$\int_{0}^{\infty} \frac{f(ax) - f(bx)}{x} \mathrm{d}x = (f(0) - f(\infty)) \ln \frac{b}{a} \tag{11}$$

We see that $b = 4$, and $a = 1$, and so the answer is $\ln \frac{b}{a} = \boxed{\ln(4)}$.