Quadratic Fun Problems

Justin Stevens

Problem 1. (Mandelbrot) Determine the positive integer a such that $x^8 + 5x^6 + 13x^4 + 20x^2 + 36$ is evenly divisible by $x^2 - x + a$.

Solution. Let $n(x) = x^8 + 5x^6 + 13x^4 + 20x^2 + 36$ and $d(x) = x^2 - x + a$, so

$$n(x) = d(x)q(x) \text{ for } q(x) \in \mathbb{Z}[x].$$

Since a is the constant term of $d(x)$, it must divide 36. Substituting $x = 1$ into the equation,

$$n(1) = d(1) \cdot q(1) \implies 75 = a \cdot q(1).$$

Since $a \mid 75$, $a \mid 36$, and a is positive, $a = 1$ or $a = 3$. Substituting $x = -2$ gives

$$n(-2) = d(-2) \cdot q(-2) \implies 900 = (a + 6) \cdot q(-2).$$

Since $7 \nmid 900$, we must have $a = \mathbf{3}$. Using long division, we verify

$$x^8 + 5x^6 + 13x^4 + 20x^2 + 36 = \left(x^2 - x + 3\right)\left(x^6 + x^5 + 3x^4 + 4x^2 + 4x + 12\right). \qquad \square$$

Problem 2. (AIME) Find integer values a and b such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$. *Hint:* Use Binet's Formula.

Solution. We know φ and ψ are the roots of $x^2 - x - 1$. Since $\varphi^2 = \varphi + 1$, we see that

$$\begin{aligned}
\varphi^3 &= \varphi^2 + \varphi = 2\varphi + 1 \\
\varphi^4 &= \varphi^3 + \varphi^2 = 3\varphi + 2 \\
\varphi^5 &= \varphi^4 + \varphi^3 = 5\varphi + 3.
\end{aligned}$$

These coefficients are Fibonacci numbers! We can prove by induction that $\varphi^n = F_n\varphi + F_{n-1}$. Since $x^2 - x - 1$ divides $ax^{17} + bx^{16} + 1$, φ must also be a root of $ax^{17} + bx^{16} + 1$. Therefore,

$$a\varphi^{17} + b\varphi^{16} + 1 = a\left(F_{17}\varphi + F_{16}\right) + b\left(F_{16}\varphi + F_{15}\right) + 1 = 0.$$

Hence, $aF_{17} + bF_{16} = 0$ and $aF_{16} + bF_{15} + 1 = 0$. We solve $a = F_{16} = \mathbf{987}$ and $b = -F_{17} = \mathbf{-1597}$. We can verify the second equation by Cassini's identity: $F_{16}^2 - F_{15}F_{17} = -1$. $\square$

Problem 3. Prove that the harmonic sum

$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

is never an integer for $n \geq 2$. *Hint:* Consider the 2-adic valuation.

Solution. Let $a_j = n!/j$ for $1 \le j \le n$, hence we can rewrite the sum as

$$\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n!}.$$

Let the largest power of 2 less than or equal to n be $r = 2^s$. Hence,

$$\begin{aligned} v_2\left(a_1 + a_2 + \cdots + a_n\right) &= v_2(a_r) \\ &= v_2(n!) - v_2(r) \\ &< v_2(n!). \end{aligned}$$

There are more factors of 2 in the denominator than numerator, so H_n is never an integer. □

Problem 4. (Cassini's Identity) Prove that $F_{n+1}F_{n-1} - F_n^2 = (-1)^2$.

Solution. Notice $\varphi\psi = -1$ and $\varphi - \psi = \sqrt{5}$. Using Binet's formula and manipulation,

$$\begin{aligned} F_{n+1}F_{n-1} - F_n^2 &= \frac{1}{5}\left[\left(\varphi^{n+1} - \psi^{n+1}\right)\left(\varphi^{n-1} - \psi^{n-1}\right) - \left(\varphi^n - \psi^n\right)^2\right] \\ &= \frac{1}{5}\left[-\varphi^{n+1}\psi^{n-1} - \psi^{n+1}\varphi^{n-1} + 2\varphi^n\psi^n\right] \\ &= -\frac{1}{5}\left(\varphi\psi\right)^{n-1}\left(\varphi^2 - 2\varphi\psi + \psi^2\right) \\ &= -\frac{1}{5}\left(-1\right)^{n-1}\left(\varphi - \psi\right)^2 \\ &= \left(-1\right)^n. \end{aligned}$$

□

Problem 5. Prove that 60 divides xyz for a Pythagorean triple (x, y, z).

Solution. Since $60 = 3 \cdot 4 \cdot 5$, we examine the equation $a^2 + b^2 = c^2$ modulo 3, 4, and 5.

We use the method of proof by contradiction three times:

- Assume that $3 \nmid abc$. The quadratic residues mod 3 are 0 and 1, so

  $$c^2 \equiv a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod{3}.$$

  However, 2 is not a quadratic residue mod 3, contradiction. Therefore, $3 \mid abc$.

- Assume that $5 \nmid abc$. The quadratic residues mod 5 are $0, 1$, and 4 so either

  $$c^2 \equiv a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod{5}, \qquad c^2 \equiv a^2 + b^2 \equiv 4 + 4 \equiv 3 \pmod{5}.$$

  However, 2 nor 3 is a quadratic residue mod 5, contradiction. Therefore, $5 \mid abc$.

- Assume that $4 \nmid abc$. If a and b are both odd, then $c^2 \equiv a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod{4}$, a nonresidue. WLOG let a be even. The quadratic residues mod 8 are $0, 1$, and 4, so

  $$a^2 \equiv c^2 - b^2 \equiv 1 - 1 \equiv 0 \pmod{8}.$$

  Therefore, $4 \mid a$. If b and c are even, we still have $4 \mid abc$.

Since we have proven the statement for $3, 5$, and 4, we conclude that $60 \mid abc$. $\square$

Problem 6. Prove the inradius of a Pythagorean triple is always an integer.

Solution. Let the triple be $\{x, y, z\} = \{2kmn, k(m^2 - n^2), k(m^2 + n^2)\}$ for an integer k. We compute the area of the triangle in two ways: $1/2r(x + y + z) = 1/2xy$. Substituting gives

$$r \cdot k(2mn + m^2 - n^2 + m^2 + n^2) = (2kmn) \cdot (k(m^2 - n^2)).$$

Simplifying, $r \cdot 2km(m + n) = 2kmn(m + n) \cdot [k(m - n)]$, so $r = k(m - n)$, an integer. $\square$