Quadratic Fun Problems

Justin Stevens

Problem 1. (Mandelbrot) Determine the positive integer $a$ such that $x^8 + 5x^6 + 13x^4 + 20x^2 + 36$ is evenly divisible by $x^2 - x + a$.

Solution. Let $n(x) = x^8 + 5x^6 + 13x^4 + 20x^2 + 36$ and $d(x) = x^2 - x + a$, so

$$n(x) = d(x)q(x) \text{ for } q(x) \in \mathbb{Z}[x].$$

Since $a$ is the constant term of $d(x)$, it must divide 36. Substituting $x = 1$ into the equation,

$$n(1) = d(1) \cdot q(1) \implies 75 = a \cdot q(1).$$

Since $a \mid 75$, $a \mid 36$, and $a$ is positive, $a = 1$ or $a = 3$. Substituting $x = -2$ gives

$$n(-2) = d(-2) \cdot q(-2) \implies 900 = (a + 6) \cdot q(-2).$$

Since $7 \nmid 900$, we must have $a = 3$. Using long division, we verify

$$x^8 + 5x^6 + 13x^4 + 20x^2 + 36 = (x^2 - x + 3) (x^6 + x^5 + 3x^4 + 4x^2 + 4x + 12).$$

Problem 2. (AIME) Find integer values $a$ and $b$ such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$. Hint: Use Binet’s Formula.

Solution. We know $\varphi$ and $\psi$ are the roots of $x^2 - x - 1$. Since $\varphi^2 = \varphi + 1$, we see that

$$\varphi^3 = \varphi^2 + \varphi = 2\varphi + 1,$$

$$\varphi^4 = \varphi^3 + \varphi^2 = 3\varphi + 2,$$

$$\varphi^5 = \varphi^4 + \varphi^3 = 5\varphi + 3.$$  

These coefficients are Fibonacci numbers! We can prove by induction that $\varphi^n = F_n\varphi + F_{n-1}$. Since $x^2 - x - 1$ divides $ax^{17} + bx^{16} + 1$, $\varphi$ must also be a root of $ax^{17} + bx^{16} + 1$. Therefore,

$$a\varphi^{17} + b\varphi^{16} + 1 = a (F_{17}\varphi + F_{16}) + b (F_{16}\varphi + F_{15}) + 1 = 0.$$

Hence, $aF_{17} + bF_{16} = 0$ and $aF_{16} + bF_{15} + 1 = 0$. We solve $a = F_{16} = 987$ and $b = -F_{17} = -1597$. We can verify the second equation by Cassini’s identity: $F_{16}^2 - F_{15}F_{17} = -1$.  

Problem 3. Prove that the harmonic sum

$$H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

is never an integer for $n \geq 2$. Hint: Consider the 2-adic valuation.
Solution. Let \( a_j = n! / j \) for \( 1 \leq j \leq n \), hence we can rewrite the sum as
\[
\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n!}.
\]
Let the largest power of 2 less than or equal to \( n \) be \( r = 2^s \). Hence,
\[
v_2(a_1 + a_2 + \cdots + a_n) = v_2(a_r) = v_2(n!) - v_2(r) < v_2(n!).
\]
There are more factors of 2 in the denominator than numerator, so \( H_n \) is never an integer.

Problem 4. (Cassini’s Identity) Prove that \( F_{n+1}F_{n-1} - F_n^2 = (-1)^n \).

Solution. Notice \( \varphi \psi = -1 \) and \( \varphi - \psi = \sqrt{5} \). Using Binet’s formula and manipulation,
\[
F_{n+1}F_{n-1} - F_n^2 = \frac{1}{5} \left[ (\varphi^{n+1} - \psi^{n+1}) (\varphi^{n-1} - \psi^{n-1}) - (\varphi^n - \psi^n)^2 \right]
\]
\[
= \frac{1}{5} \left[ -\varphi^{n+1} \psi^{n-1} + \psi^{n+1} \varphi^{n-1} + 2\varphi^n \psi^n \right]
\]
\[
= -\frac{1}{5} (\varphi \psi)^{n-1} (\varphi^2 - 2\varphi \psi + \psi^2)
\]
\[
= -\frac{1}{5} (-1)^{n-1} (\varphi - \psi)^2
\]
\[
= (-1)^n.
\]

Problem 5. Prove that 60 divides \( xyz \) for a Pythagorean triple \((x, y, z)\).

Solution. Since 60 = 3 · 4 · 5, we examine the equation \( a^2 + b^2 = c^2 \) modulo 3, 4, and 5.

We use the method of proof by contradiction three times:

- Assume that 3 \( \nmid \) \( abc \). The quadratic residues mod 3 are 0 and 1, so
\[
c^2 \equiv a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod{3}.
\]
However, 2 is not a quadratic residue mod 3, contradiction. Therefore, 3 \( \mid \) \( abc \).

- Assume that 5 \( \nmid \) \( abc \). The quadratic residues mod 5 are 0, 1, and 4 so either
\[
c^2 \equiv a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod{5}, \quad c^2 \equiv a^2 + b^2 \equiv 4 + 4 \equiv 3 \pmod{5}.
\]
However, 2 nor 3 is a quadratic residue mod 5, contradiction. Therefore, 5 \( \mid \) \( abc \).

- Assume that 4 \( \nmid \) \( abc \). If \( a \) and \( b \) are both odd, then \( c^2 \equiv a^2 + b^2 \equiv 1 + 1 \equiv 2 \pmod{4} \), a nonresidue. WLOG let \( a \) be even. The quadratic residues mod 8 are 0, 1, and 4, so
\[
a^2 \equiv c^2 - b^2 \equiv 1 - 1 \equiv 0 \pmod{8}.
\]
Therefore, 4 \( \mid a \). If \( b \) and \( c \) are even, we still have 4 \( \mid \) \( abc \).
Since we have proven the statement for 3, 5, and 4, we conclude that $60 \mid abc$.

**Problem 6.** Prove the inradius of a Pythagorean triple is always an integer.

**Solution.** Let the triple be $\{x, y, z\} = \{2kmn, k(m^2 - n^2), k(m^2 + n^2)\}$ for an integer $k$. We compute the area of the triangle in two ways: $1/2r(x + y + z) = 1/2xy$. Substituting gives

$$r \cdot k(2mn + m^2 - n^2 + m^2 + n^2) = (2kmn) \cdot (k(m^2 - n^2)).$$

Simplifying, $r \cdot 2km(m + n) = 2kmn(m + n) \cdot [k(m - n)]$, so $r = k(m - n)$, an integer.