Multiplicative Function Solutions

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Problem 1. Calculate the sum of the positive divisors of 1440 that are divisible by 6.

Solution. Since $1440 = 2^5 \cdot 3^2 \cdot 5$, every divisor that is divisible by 6 is given in the product

$$\left(2 + 2^2 + 2^3 + 2^4 + 2^5\right)\left(3 + 3^2\right)(1 + 5) = 62 \cdot 12 \cdot 6 = \boxed{4464}. \qquad \square$$

Problem 2. Solve the linear congruence $140x \equiv 133 \pmod{301}$.

Solution. Observe that $\gcd(140, 301) = 7$. Therefore, we cancel the factor to see

$$20x \equiv 19 \pmod{43} \implies 20x = 43m + 19 \text{ for } m \in \mathbb{Z}.$$

Taking this equation modulo 20, we see $3m \equiv 1 \pmod{20} \implies m \equiv 7 \pmod{20}$. Substituting, we see $20x = 43 \cdot 7 + 19 \implies x = 16$. Therefore, the solution is $x \equiv \boxed{16 \pmod{43}}$. $\square$

Problem 3. (AIME) How many positive integers are divisors of at least one of $10^{10}, 15^7, 18^{11}$?

Solution. We factore $10^{10} = 2^{10}5^{10}$, so it has $11 \cdot 11 = 121$ divisors. Furthermore $15^7 = 3^7 \cdot 5^7$, so it has $8 \cdot 8 = 64$ divisors, and $18^{11} = 2^{11} \cdot 3^{22}$, so it has $12 \cdot 23 = 276$ divisors.

We now have to subtract off divisors of two numbers by Principle of Inclusion-Exclusion. $\gcd(10^{10}, 15^7) = 5^7$ has 8 divisors,$\gcd(10^{10}, 18^{11}) = 2^{10}$ has 11, and $\gcd(15^7, 18^{11}) = 3^7$ has 8.

Finally, we add back divisors all three of our numbers, namely 1. Hence, our answer is

$$121 + 64 + 276 - 8 - 11 - 8 + 1 = \boxed{435}. \qquad \square$$

Problem 4. (AIME) Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digits, then $p(n)$ is equal to that digit.) Let

$$S = p(1) + p(2) + p(3) + \cdots + p(999).$$

What is the largest prime factor of S?

Solution. The numbers 0 to 999 can be expressed as abc where $0 \leq a, b, c \leq 9$. Consider

$$(1 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)(1 + 1 + \cdots + 9)(1 + 1 + \cdots + 9).$$

The first 1 replaces the digit zero. The product of the non-zero digits of every number is represented in this expression. To find S, we subtract $p(0) = 1$:

$$S = 46^3 - 1 = (46 - 1)\left(46^2 + 46 + 1\right) = 3^3 \cdot 5 \cdot 7 \cdot 103.$$

The largest prime factor of S is hence $\boxed{103}$. $\square$

Problem 5. (India TST) On the real number line, paint red all points that correspond to integers of the form $81x+100y$, where x and y are nonnegative integers. Paint the remaining integer points blue. Find with proof a point P on the line such that, for every integer point T, the reflection of T with respect to P is an integer point of a different colour than T.

Solution. The answer is $g(81, 100)/2 = \boxed{3959.5}$. In general, $P = g(a, b)/2$. Clearly, all the red points are the representable integers and the blue points are the non-representable integers. Assume for the sake of contradiction that both t and $g(a, b) - t$ are red. Therefore,

$$\begin{aligned} ax_1 + by_1 &= t \\ ax_2 + by_2 &= g(a, b) - t. \end{aligned}$$

However, adding these two equations gives $a(x_1+x_2)+b(y_1+y_2) = g(a, b)$, contradiction. Considering the pairs $(t, g(a, b) - t)$ for $0 \le t \le (g(a, b) - 1)/2$, at most one element is red. Furthermore, we showed in class that there are $(a-1)(b-1)/2 = (g(a, b)+1)/2$ blue numbers. Hence, every pair must consist of exactly one red number and one blue number. □

Problem 6. The Liouville function is defined by $\lambda(n) = (-1)^{\Omega(n)}$. Prove that.

(a) $\lambda(n)$ is completely multiplicative.

(b) $\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if n is a perfect square} \\ 0 & \text{otherwise} \end{cases}$.

Solution. Recall that $\Omega(n) = \sum_{p|n} v_p(n)$ is the number of prime factors function.

(a) We see Ω is an additive function, so $\Omega(mn) = \Omega(m) + \Omega(n)$ for all m and n Hence,

$$\lambda(mn) = (-1)^{\Omega(mn)} = (-1)^{\Omega(m)+\Omega(n)} = (-1)^{\Omega(m)} (-1)^{\Omega(n)} = \lambda(m)\lambda(n).$$

(b) Since λ is multiplicative, we only need to check the result for prime powers $n = p^k$:

$$\begin{aligned} \sum_{d|p^k} \lambda(d) &= \lambda(1) + \lambda(p) + \lambda(p^2) + \cdots + \lambda(p^k) \\ &= \sum_{0\le i\le k} (-1)^i \\ &= \begin{cases} 1 & \text{if k is even} \\ 0 & \text{if k is odd.} \end{cases} \end{aligned}$$

The conclusion that the exponent is equivalent is equivalent to n being a perfect square.

□