Primes and Polynomials Solutions

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Problem 1. (Math Prize for Girls) Find the unique five-digit prime divisor of $104,060,465$.

Solution. Observe that $104,060,465 = 104060401 + 64$. By the binomial theorem, we have $104060401 = 101^4$. Therefore, using the Sophie-Germain identity,

$$\begin{aligned}104060465 &= 101^4 + 64 = 101^4 + 4 \cdot 2^4 \\ &= \left(101^2 + 2 \cdot 2^2 - 2 \cdot 101 \cdot 2\right)\left(101^2 + 2 \cdot 2^2 + 2 \cdot 101 \cdot 2\right) \\ &= 9805 \cdot 10613.\end{aligned}$$

The answer is hence $\boxed{10613}$. □

Problem 2. (AMC 10) Find the number of ordered triples (x, y, z) of positive integers that satisfy $\operatorname{lcm}(x, y) = 72, \operatorname{lcm}(x, z) = 600$, and $\operatorname{lcm}(y, z) = 900$.

Solution. Prime factorizing we see that $72 = 2^3 \cdot 3^2$, $600 = 2^3 \cdot 3 \cdot 5^2$, and $900 = 2^2 \cdot 3^2 \cdot 5^2$. We consider each prime factor separately.

- For the factor of 2, we see that we must have $v_2(x) = 3$. We furthermore must have $\max(v_2(y), v_2(z)) = 2$, giving the pairs $(2,0),(2,1),(2,2),(0,2),(1,2)$ for **5** pairs.
- For the factor of 3, we must have $v_2(y) = 2$. We furthermore must have $\max(v_2(x), v_2(z)) = 1$, giving the pairs $(1,0),(0,1),(1,1)$ for a total of **3** pairs.
- For the factor of 5, we must have $v_2(z) = 2$, $v_2(x) = 0$, and $v_2(y) = 0$ for **1** pair.

Hence, by the product rule, the answer is $5 \cdot 3 \cdot 1 = \boxed{15}$. □

Problem 3. (AHSME) For how many integers N between 1 and 1990 is the improper fraction $\frac{N^2+7}{N+4}$ not in lowest terms?

Solution. Using the Euclidean algorithm, since $N^2 - 16 = (N+4)(N-4)$, we have that

$$\gcd(N^2 + 7, N + 4) = \gcd(N^2 + 7 - (N^2 - 16), N + 4) = \gcd(23, N + 4).$$

Therefore, for the fraction to be reducible, we must have $N \equiv 19 \pmod{23}$. The numbers between 1 and 1990 that are $19 \pmod{23}$ are $\{19, 42, 65, \cdots, 1974\}$. Since $19 = 23 \cdot 0 + 19$ and $1974 = 23 \cdot 85 + 19$, there are $\boxed{86}$ integers. □

Problem 4. If a and b are integers such that $x^2 - x - 1$ is a factor of $ax^3 + bx^2 + 1$, then what are the values of a and b?

Solution. We use the division algorithm. First off, divide the leading terms: $\frac{ax^3}{x^2} = ax$.

$$ax^3 + bx^2 + 1 = \left(x^2 - x - 1\right)(ax) + \left[(a+b)x^2 + ax + 1\right].$$

To eliminate the quadratic term from the remainder, we add $a + b$:

$$ax^3 + bx^2 + 1 = \left(x^2 - x - 1\right)(ax + a + b) + \left[(2a+b)x + a + b + 1\right].$$

In order for $x^2 - x - 1$ to divide $ax^3 + bx^2 + 1$, the remainder must be 0. Therefore we equate the coefficients to be zero:

$$\begin{cases} 2a + b = 0 \\ a + b + 1 = 0. \end{cases} \quad \Longrightarrow (a, b) = \boxed{(1, -2)}$$

We can indeed verify that $x^3 - 2x^2 + 1 = (x^2 - x - 1)(x - 1)$. □

Problem 5. Prove that there are infinitely many primes of the form $4k + 3$.

Hint: The product of two numbers that are 1 mod 4 is also 1 mod 4.

Solution. Assume for the sake of contradiction that there are finite primes of the form $4k+3$, namely $\{p_1, p_2, \cdots, p_k\}$. Consider the number $N = 4p_1p_2 \cdots p_k - 1$. Since $N \equiv 3 \pmod 4$ and the product of numbers of the form 1 mod 4 is also 1 mod 4, N must have a prime divisor that is of the form $3 \pmod 4$, say q. Since $p_i \nmid N$ for $1 \le i \le k$, q is another prime outside of the set. Hence, we have constructed a new prime number, contradicting the finality of the number of primes. The conclusion is that there are infinite primes of the form $4k + 3$. □

Problem 6. The symbols $(a, b, \ldots, g)$ and $[a, b, \ldots, g]$ denote the greatest common divisor and least common multiple, respectively, of the positive integers $a, b, \ldots, g$. For example, $(3, 6, 18) = 3$ and $[6, 15] = 30$. Prove that

$$\frac{[a, b, c]^2}{[a, b][b, c][c, a]} = \frac{(a, b, c)^2}{(a, b)(b, c)(c, a)}.$$

Hint: Consider the number of factors of a prime p in each side of the expression.

Solution. Consider a prime p dividing at least one of a, b, c. Let $v_p(a) = \alpha, v_p(b) = \beta$, and $v_p(c) = \gamma$. WLOG let $\alpha \ge \beta \ge \gamma$. Then, the number of factors of p on the left hand side is

$$v_p(\frac{[a, b, c]^2}{[a, b][b, c][c, a]}) = 2v_p([a, b, c]) - v_p([a, b]) - v_p([b, c]) - v_p([c, a]) = 2\alpha - \alpha - \beta - \alpha = -\beta.$$

Furthermore, the number of factors of p on the right hand side is

$$v_p(\frac{(a, b, c)^2}{(a, b)(b, c)(c, a)}) = 2v_p\left((a, b, c)\right) - v_p\left((a, b)\right) - v_p\left((b, c)\right) - v_p\left((c, a)\right) = 2\gamma - \beta - \gamma - \gamma = -\beta.$$

Therefore, the number of factors of any prime on both sides are equivalent. By the Fundamental Theorem of Arithmetic, this implies that both sides of the equation are equal. □