Primes and Polynomials Solutions

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**Problem 1.** (Math Prize for Girls) Find the unique five-digit prime divisor of 104,060,465.

*Solution.* Observe that 104,060,465 = 104,060,401 + 64. By the binomial theorem, we have 104,060,401 = 101^4. Therefore, using the Sophie-Germain identity,

\[104,060,465 = 101^4 + 64 = 101^4 + 4 \cdot 2^4\]

\[= (101^2 + 2 \cdot 2^2 - 2 \cdot 101 \cdot 2) (101^2 + 2 \cdot 2^2 + 2 \cdot 101 \cdot 2)\]

\[= 9805 \cdot 10613.\]

The answer is hence \(10613\).

**Problem 2.** (AMC 10) Find the number of ordered triples \((x,y,z)\) of positive integers that satisfy lcm\((x,y)\) = 72, lcm\((x,z)\) = 600, and lcm\((y,z)\) = 900.

*Solution.* Prime factorizing we see that 72 = 2^3 \cdot 3^2, 600 = 2^3 \cdot 3^2 \cdot 5^2, and 900 = 2^2 \cdot 3^2 \cdot 5^2. We consider each prime factor separately.

- For the factor of 2, we see that we must have \(v_2(x) = 3\). We furthermore must have \(\max(v_2(y), v_2(z)) = 2\), giving the pairs (2,0), (2,1), (2,2), (0,2), (1,2) for 5 pairs.
- For the factor of 3, we must have \(v_3(y) = 2\). We furthermore must have \(\max(v_3(x), v_3(z)) = 1\), giving the pairs (1,0), (0,1), (1,1) for a total of 3 pairs.
- For the factor of 5, we must have \(v_5(z) = 2\), \(v_5(x) = 0\), and \(v_5(y) = 0\) for 1 pair.

Hence, by the product rule, the answer is 5 \cdot 3 \cdot 1 = 15.

**Problem 3.** (AHSME) For how many integers \(N\) between 1 and 1990 is the improper fraction \(\frac{N^2 + 7}{N + 4}\) not in lowest terms?

*Solution.* Using the Euclidean algorithm, since \(N^2 - 16 = (N + 4)(N - 4)\), we have that

\[\gcd(N^2 + 7, N + 4) = \gcd(N^2 + 7 - (N^2 - 16), N + 4) = \gcd(23, N + 4).\]

Therefore, for the fraction to be reducible, we must have \(N \equiv 19 \pmod{23}\). The numbers between 1 and 1990 that are 19 \((\mod{23})\) are \{19, 42, 65, \cdots, 1974\}. Since 19 = 23 \cdot 0 + 19 and 1974 = 23 \cdot 85 + 19, there are 86 integers.

**Problem 4.** If \(a\) and \(b\) are integers such that \(x^2 - x - 1\) is a factor of \(ax^3 + bx^2 + 1\), then what are the values of \(a\) and \(b\)?
Solution. We use the division algorithm. First off, divide the leading terms: \( \frac{ax^3}{x^2} = ax \).

\[
ax^3 + bx^2 + 1 = (x^2 - x - 1) (ax) + \left[ (a + b)x^2 + ax + 1 \right].
\]

To eliminate the quadratic term from the remainder, we add \( a + b \):

\[
ax^3 + bx^2 + 1 = (x^2 - x - 1) (ax + a + b) + [(2a + b)x + a + b + 1].
\]

In order for \( x^2 - x - 1 \) to divide \( ax^3 + bx^2 + 1 \), the remainder must be 0. Therefore we equate the coefficients to be zero:

\[
\begin{align*}
2a + b &= 0 \\
\Rightarrow (a, b) &= (1, -2)
\end{align*}
\]

We can indeed verify that \( x^3 - 2x^2 + 1 = (x^2 - x - 1) (x - 1) \).

Problem 5. Prove that there are infinitely many primes of the form \( 4k + 3 \).

Hint: The product of two numbers that are 1 mod 4 is also 1 mod 4.

Solution. Assume for the sake of contradiction that there are finite primes of the form \( 4k + 3 \), namely \( \{ p_1, p_2, \ldots, p_k \} \). Consider the number \( N = 4p_1p_2 \cdots p_k - 1 \). Since \( N \equiv 3 \) (mod 4) and the product of numbers of the form 1 mod 4 is also 1 mod 4, \( N \) must have a prime divisor that is of the form 3 (mod 4), say \( q \). Since \( p_i \nmid N \) for \( 1 \leq i \leq k \), \( q \) is another prime outside of the set. Hence, we have constructed a new prime number, contradicting the finiteness of the number of primes. The conclusion is that there are infinite primes of the form \( 4k + 3 \).

Problem 6. The symbols \( (a, b, \ldots, g) \) and \( [a, b, \ldots, g] \) denote the greatest common divisor and least common multiple, respectively, of the positive integers \( a, b, \ldots, g \). For example, \( (3, 6, 18) = 3 \) and \( [6, 15] = 30 \). Prove that

\[
\frac{[a, b, c]^2}{[a, b][b, c][c, a]} = \frac{(a, b, c)^2}{(a, b)(b, c)(c, a)}.
\]

Hint: Consider the number of factors of a prime \( p \) in each side of the expression.

Solution. Consider a prime \( p \) dividing at least one of \( a, b, c \). Let \( v_p(a) = \alpha, v_p(b) = \beta, \) and \( v_p(c) = \gamma \). WLOG let \( \alpha \geq \beta \geq \gamma \). Then, the number of factors of \( p \) on the left hand side is

\[
v_p \left( \frac{[a, b, c]^2}{[a, b][b, c][c, a]} \right) = 2v_p([a, b, c]) - v_p([a, b]) - v_p([b, c]) - v_p([c, a]) = 2\alpha - \alpha - \beta - \alpha = -\beta.
\]

Furthermore, the number of factors of \( p \) on the right hand side is

\[
v_p \left( \frac{(a, b, c)^2}{(a, b)(b, c)(c, a)} \right) = 2v_p((a, b, c)) - v_p((a, b)) - v_p((b, c)) - v_p((c, a)) = 2\gamma - \beta - \gamma - \gamma = -\beta.
\]

Therefore, the number of factors of any prime on both sides are equivalent. By the Fundamental Theorem of Arithmetic, this implies that both sides of the equation are equal.