# Remainders Solutions

Justin Stevens

**Problem 1.** Compute by hand $7^{50} \pmod{43}$.

*Solution.* Notice that $7^3 = 343 \equiv -1 \pmod{43}$, since $344 = 43 \cdot 8$. Hence,

$$7^{50} \equiv \left(7^3\right)^{16} \cdot 7^2 \equiv 1^{16} \cdot 49 \equiv 6 \pmod{43}.$$

□

**Problem 2.** Compute the last three digits of $2011^{2011}$.

*Solution.* Notice that $2011^{2011} \equiv 11^{2011}$. Using the binomial theorem, we see that

$$\begin{aligned}
11^{2011} &= (1+10)^{2011} \\
&\equiv 1^{2011} + \binom{2011}{1} \cdot 1^{2010} \cdot 10 + \binom{2011}{2} \cdot 1^{2009} \cdot 10^2 + \cdots \\
&\equiv 1 + 11 \cdot 1 \cdot 10 + \frac{2011 \cdot 2010}{2} \cdot 1 \cdot 100 \\
&\equiv 1 + 110 + 11 \cdot 5 \cdot 100 \\
&\equiv \boxed{661} \pmod{1000}.
\end{aligned}$$

□

**Problem 3.** Given that 1002004008016032 has a prime factor $p > 250000$, find it by hand.

*Solution.* Observe that if $x = 1000$ and $y = 2$, then the given number is simply $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$. By our difference of powers factorization,

$$\begin{aligned}
x^5 + x^4y + x^3y^2 + x^2y^3 + xy^2 + y^5 &= \frac{x^6 - y^6}{x - y} \\
&= \frac{1000^6 - 2^6}{1000 - 2} \\
&= 2^5 \cdot \frac{500^6 - 1}{500 - 1} \\
&= 2^5 \cdot \frac{(500^3 - 1)(500^3 + 1)}{500 - 1} \\
&= 2^5 \cdot \frac{(500 - 1)(500^2 + 500 + 1)(500 + 1)(500^2 - 500 + 1)}{500 - 1}.
\end{aligned}$$

The only prime factor of this number that satisfies $p > 250000$ is $500^2 + 500 + 1 = \boxed{250501}$. □

**Problem 4.** Let $S$ be a subset of $\{1, 2, 3, ..., 50\}$ such that no pair of distinct elements in $S$ has a sum divisible by 7. What is the maximum number of elements in $S$?

*Solution.* We group the numbers in the list into their mod 7 residues:

$$\begin{aligned}
0 \quad &\text{mod } 7 : \{7, 14, 21, 28, 35, 42, 49\} \\
1 \quad &\text{mod } 7 : \{1, 8, 15, 22, 29, 36, 43, 50\} \\
2 \quad &\text{mod } 7 : \{2, 9, 16, 23, 30, 37, 44\} \\
3 \quad &\text{mod } 7 : \{3, 10, 17, 24, 31, 38, 45\} \\
4 \quad &\text{mod } 7 : \{4, 11, 18, 25, 32, 39, 46\} \\
5 \quad &\text{mod } 7 : \{5, 12, 19, 26, 33, 40, 47\} \\
6 \quad &\text{mod } 7 : \{6, 13, 20, 27, 34, 41, 48\}.
\end{aligned}$$

We can have a maximum of **1** number that is $0 \pmod 7$. We can have **8** numbers that are $1 \pmod 7$, **7** numbers that are $2 \pmod 7$, and **7** numbers that are $3 \pmod 7$. Hence, we can have a maximum of $1 + 8 + 7 + 7 = \boxed{23}$. □

**Problem 5.** Use the definition of mods to prove the following statements.

(i) If $m_1 \equiv m_2 \pmod d$ and $n_1 \equiv n_2 \pmod d$, then $m_1 + n_1 \equiv m_2 + n_2 \pmod d$.

(ii) If $m_1 \equiv m_2 \pmod d$ and $n_1 \equiv n_2 \pmod d$, then $m_1 n_1 \equiv m_2 n_2 \pmod d$.

*Solution.* (i) From the definition of modulos,

$$\begin{aligned}
m_1 \equiv m_2 \pmod d &\implies d \mid m_1 - m_2 \\
n_1 \equiv n_2 \pmod d &\implies d \mid n_1 - n_2.
\end{aligned}$$

From our dividing a sum result, $d \mid (m_1 + n_1) - (m_2 + n_2)$.

Therefore, $m_1 + n_1 \equiv m_2 + n_2 \pmod d$.

(ii) Using the division algorithm, the given conditions imply there exist integers $q_m$ and $q_n$ such that

$$\begin{aligned}
m_1 &= dq_m + m_2 \\
n_1 &= dq_n + n_2.
\end{aligned}$$

Multiplying these equations shows that

$$\begin{aligned}
m_1 n_1 &\equiv (dq_m + m_2)(dq_n + n_2) \\
&\equiv d^2 q_m q_n + dq_m n_2 + dq_n m_2 + m_2 n_2 \\
&\equiv m_2 n_2 \pmod d.
\end{aligned}$$

**Comment:** Note that we alternatively could have finished part (i) by observing that

$$m_1 + n_1 = dq_m + dq_n + m_2 + n_2 \equiv m_2 + n_2 \pmod d.$$

□

**Problem 6.** (i) Show that 31 and 127 both divide $2^{35} - 1$.

(ii) Prove that if $m \mid n$, then $x^m - y^m \mid x^n - y^n$.

(iii) Prove that the number $1\underbrace{0\cdots0}_{800\text{ 0's}}1$ is divisible by 1001.

*Solution.* (i) Observe that $31 = 2^5 - 1$ and $127 = 2^7 - 1$. Now, we use the fact that $x - 1 \mid x^m - 1$ with $x = 2^5$ and $m = 7$ and $x = 2^7$ and $m = 5$.

(ii) Write $n = qm$ for some integer $q$. Then,

$$x^n - y^n = x^{qm} - y^{qm} = (x^m)^q - (y^m)^q = (x^m - y^m)\left(x^{m\cdot(q-1)} + x^{m\cdot(q-2)} \cdot y + \cdots + y^{m\cdot(q-1)}\right).$$

Hence, $x^m - y^m \mid x^n - y^n$.

(iii) The number is equivalent to $10^{801} + 1$. Observe that since $10^3 \equiv -1 \pmod{1001}$ that we have

$$10^{801} + 1 \equiv (-1)^{267} + 1 \equiv -1 + 1 \equiv 0 \pmod{1001}.$$

□

★ **Problem 7.** Prove using induction that for all positive integers $n$, $2^{2^n} + 3^{2^n} + 5^{2^n}$ is divisible by 19.