Number Theory with python PROGRAMMING

Harmonic Sums

Lecture 11

Justin Stevens

Outline

Divergent Harmonic Series

Definition. The **harmonic series** is the divergent infinite series:

$$\sum_{k=1}^{\infty} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots .$$

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One way to prove this series is divergent is the comparison test:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots$$

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Ant Traveling on Rubber Band Paradox

Example. Suppose an ant crawls along an infinitely-elastic one-meter rubber band at the same time as the rubber band is uniformly stretched. If the ant travels 1 centimeter per minute and the band stretches 1 meter per minute, will the ant ever reach the end of the rubber band?

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Time	Length of Band	Distance Traveled	Ratio Traveled
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After n seconds, the ratio of distance traveled by the ant to total length is

$$\frac{1}{100}\sum_{k=1}^{n}\frac{1}{k}.$$

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$(b^m)^n = b^{mn}$	$\log_b(x^y) = y \log_b(x)$

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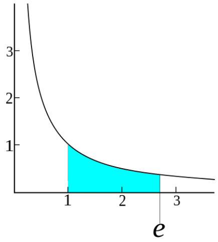

Figure 1: e is the unique number such that the shaded area of $y = 1/x$ equals 1.

Euler-Mascheroni Constant

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- In 1898, de la Vallée-Poussin proved $\gamma = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} (\lceil \frac{n}{k} \rceil - \frac{n}{k})$.

Euler-Mascheroni Constant

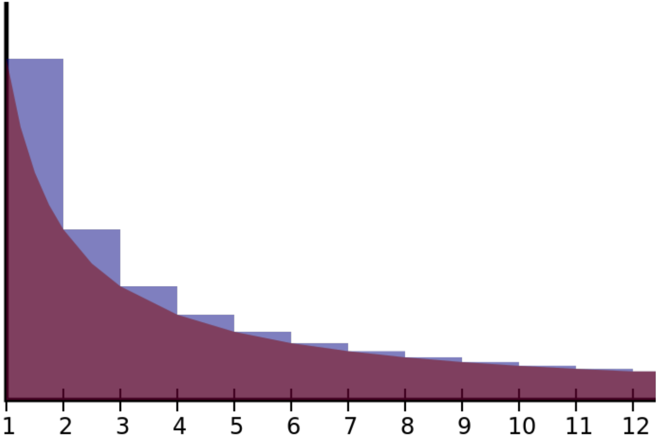

Figure 2: The area of the blue region converges to the Euler–Mascheroni constant. Source: William Demchick

How Euler Did It

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Theorem. The infinite product $\frac{2\cdot3\cdot5\cdot7\cdot11\cdot13\cdot17\cdots}{1\cdot2\cdot4\cdot6\cdot10\cdot12\cdot16\cdots}$ equals the sum of the divergent harmonic series, where each factor is of the form $p/(p-1)$.

Proof.

By the Fundamental Theorem of Arithmetic and distributive property,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p\in\mathbb{P}} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right)$$

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Since $|1/p|<1$, we can use the infinite geometric series formula:

$$=\prod_{p\in\mathbb{P}}\left(\frac{1}{1-1/p}\right).$$

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Sums of Reciprocals of Primes

Taking the natural logarithm of both sides, we see

$$\ln\left(\sum_{n=1}^{\infty}\frac{1}{n}\right) = \sum_{p}\ln\left(\frac{1}{1-1/p}\right)$$

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For $|x|<1$, $\frac{1}{1-x}=1+x+x^2+\cdots$. Taking the definite integral,

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Therefore, $\ln\left(\sum_{n=1}^{\infty}\frac{1}{n}\right) = \sum_{p}\left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \cdots\right).$

Divergence of Sum of Reciprocals of Primes

$$\ln\left(\sum_{n=1}^{\infty} \frac{1}{n}\right) = \sum_{p} \frac{1}{p} + \sum_{p} \frac{1}{p^2}\left(\frac{1}{2} + \frac{1}{3p} + \frac{1}{4p^2} + \frac{1}{5p^3} + \cdots\right)$$

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The value M is known as the Meissel-Mertens constant and depends on γ.

p-adic Valuation

Let a positive integer $n > 1$ be written as $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$.

Definition. For each prime, the ***p*-adic valuation** of n is $v_{p_i}(n) = e_i$.

If $\mathbb{P}$ is the set of primes, then another way to write the factorization is

$$n = \prod_{p \in \mathbb{P}} p^{v_p(n)}.$$

Theorem. $v_p(mn) = v_p(m) + v_p(n)$ and $v_p(n^c) = c v_p(n)$.

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Theorem. $v_p(m/n) = v_p(m) - v_p(n)$.

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Since $p \nmid a'$ and $v_p(b) - v_p(a) \geq 1$, we see that

$$a' + p^{v_p(b)-v_p(a)}b' \equiv a' \not\equiv 0 \pmod{p}.$$

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Proof.

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$$\begin{aligned} a + b &= p^{v_p(a)}a' + p^{v_p(b)}b' \\ &= p^{v_p(a)}\left(a' + p^{v_p(b)-v_p(a)}b'\right). \end{aligned}$$

Since $p \nmid a'$ and $v_p(b) - v_p(a) \geq 1$, we see that

$$a' + p^{v_p(b)-v_p(a)}b' \equiv a' \not\equiv 0 \pmod{p}.$$

Hence, $v_p(a+b) = v_p(a) = \min(v_p(a), v_p(b))$. □

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Since there are strictly more factors of 2 in the denominator than numerator, H_n is never an integer for $n \geq 2$.

Outline

Shuffling Decks of Cards

- For a standard deck of playing cards, there are $52! \approx 8.0658 \cdot 10^{67}$ permutations, more than the number of particles in the universe.

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	1	2	3	4	5	6	7	8	9
No cut	31.17	19.69	12.92	8.80	6.56	5.51	5.01	4.76	4.65
Cut	29.45	19.09	12.69	8.70	6.50	5.46	4.97	4.73	4.63

Table 1: Number of cards guessed correctly after k shuffles of 52 cards

Charles Jordan Premo Card Trick

A♣	2♣	3♣	4♣	5♣	6♣	7♣	8♣	9♣	10♣	J♣	Q♣	K♣
A♦	2♦	3♦	4♦	5♦	6♦	7♦	8♦	9♦	10♦	J♦	Q♦	K♦
A♥	2♥	3♥	4♥	5♥	6♥	7♥	8♥	9♥	10♥	J♥	Q♥	K♥
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1. Have a spectator cut the deck several times and give it a riffle shuffle.

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2. Repeat the first step, then turn your back to the spectator.
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4. The card is then burried by cutting the deck and riffle shuffling.
5. The performer then turns back and deals the cards into several face-up rows and after some careful thought, reveals the spectator's card.

Premo First Shuffle

The spectator begins by cutting the deck and giving it a riffle shuffle:

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K♠	A♣	2♣	J♦	Q♦	3♣	4♣	K♦	A♥	5♣	2♥	6♣	7♣
8♣	3♥	9♣	10♣	J♣	Q♣	4♥	5♥	K♣	6♥	A♦	7♥	2♦
8♥	9♥	3♦	4♦	10♥	J♥	Q♥	5♦	6♦	K♥	A♠	7♦	2♠
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8♥	9♥	3♦	4♦	10♥	J♥	Q♥	5♦	6♦	K♥	A♠	7♦	2♠
3♠	4♠	5♠	8♦	6♠	7♠	8♠	9♠	9♦	10♠	J♠	Q♠	10♦

Definition. A rising sequence is a maximal subset of an arrangement of cards, consisting of successive face values displayed in order.

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8♥	9♥	3♦	4♦	10♥	J♥	Q♥	5♦	6♦	K♥	A♠	7♦	2♠
3♠	4♠	5♠	8♦	6♠	7♠	8♠	9♠	9♦	10♠	J♠	Q♠	10♦

Definition. A rising sequence is a maximal subset of an arrangement of cards, consisting of successive face values displayed in order. Each arrangement of a deck of cards is the union of its rising sequences.

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The spectator then cuts the deck again and gives it a second riffle shuffle:

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J♥	2♥	6♣	7♣	Q♥	8♣	5♦	3♥	9♣	6♦	K♥	10♣	J♣
A♠	7♦	Q♣	2♠	3♠	4♥	4♠	5♥	5♠	K♣	8♦	6♠	6♥
7♠	8♠	9♠	9♦	A♦	10♠	J♠	Q♠	7♥	10♦	K♠	A♣	2♦

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J♥	2♥	6♣	7♣	Q♥	8♣	5♦	3♥	9♣	6♦	K♥	10♣	J♣
A♠	7♦	Q♣	2♠	3♠	4♥	4♠	5♥	5♠	K♣	8♦	6♠	6♥
7♠	8♠	9♠	9♦	A♦	10♠	J♠	Q♠	7♥	10♦	K♠	A♣	2♦

There are now the following four rising sequences:

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J♥	2♥	6♣	7♣	Q♥	8♣	5♦	3♥	9♣	6♦	K♥	10♣	J♣
A♠	7♦	Q♣	2♠	3♠	4♥	4♠	5♥	5♠	K♣	8♦	6♠	6♥
7♠	8♠	9♠	9♦	A♦	10♠	J♠	Q♠	7♥	10♦	K♠	A♣	2♦

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- 2♣ 3♣ 4♣ 5♣ 6♣ 7♣ 8♣ 9♣ 10♣ J♣ Q♣ K♣ A♦ 2♦
- J♦ Q♦ K♦ A♥ 2♥ 3♥ 4♥ 5♥ 6♥ 7♥
- 3♦ 4♦ 5♦ 6♦ 7♦ 8♦ 9♦ 10♦

Premo Card Selection

The spectator then lifts off a quarter of the cards, chooses the top card, and puts the removed card on top of the reconstructed deck. This card is then burried by cutting the deck and riffle shuffling a third and final time:

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A♦	10♠	6♣	J♠	Q♠	7♥	10♦	K♠	A♣	2♦	Q♥	8♣	5♦
7♣	8♥	3♥	9♣	2♣	J♦	6♦	K♥	9♥	3♦	Q♦	10♣	J♣
4♦	A♠	3♣	4♣	7♦	10♥	Q♣	K♦	2♠	3♠	A♥	4♥	5♣

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A♦	10♠	6♣	J♠	Q♠	7♥	10♦	K♠	A♣	2♦	Q♥	8♣	5♦
7♣	8♥	3♥	9♣	2♣	J♦	6♦	K♥	9♥	3♦	Q♦	10♣	J♣
4♦	A♠	3♣	4♣	7♦	10♥	Q♣	K♦	2♠	3♠	A♥	4♥	5♣

We can detect the following nine rising sequences:

Premo Card Selection

The spectator then lifts off a quarter of the cards, chooses the top card, and puts the removed card on top of the reconstructed deck. This card is then burried by cutting the deck and riffle shuffling a third and final time:

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A♦	10♠	6♣	J♠	Q♠	7♥	10♦	K♠	A♣	2♦	Q♥	8♣	5♦
7♣	8♥	3♥	9♣	2♣	J♦	6♦	K♥	9♥	3♦	Q♦	10♣	J♣
4♦	A♠	3♣	4♣	7♦	10♥	Q♣	K♦	2♠	3♠	A♥	4♥	5♣

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- 4♠ 5♠ 6♠ 7♠ 8♠ 9♠ 10♠ J♠ Q♠ K♠
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- K♣ A♦ 2♦ 3♦ 4♦
- 8♦ 9♦ 10♦ J♦ Q♦ K♦ A♥
- 2♥ 3♥ 4♥ 5♦ 6♦ 7♦
- A♣ 2♣ 3♣ 4♣ 5♣
- 6♣ 8♣ 9♣ 10♣ J♣ Q♣

Book Stacking for n Books

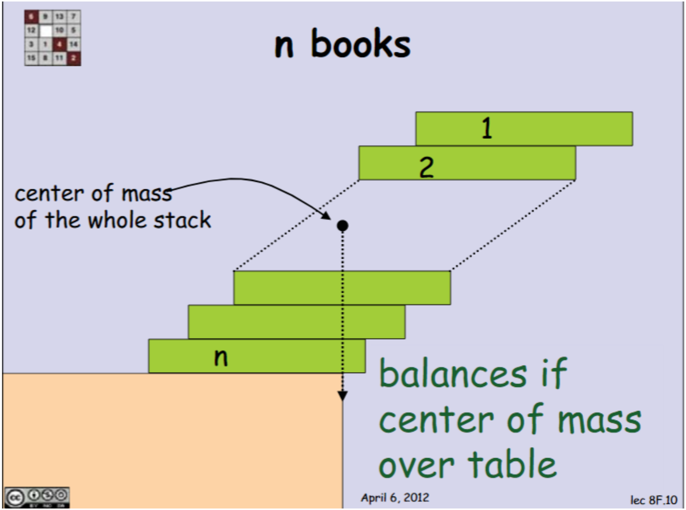

Figure 3: Albert R. Meyer, MIT 6.042J

Book Stacking for $n+1$ Books

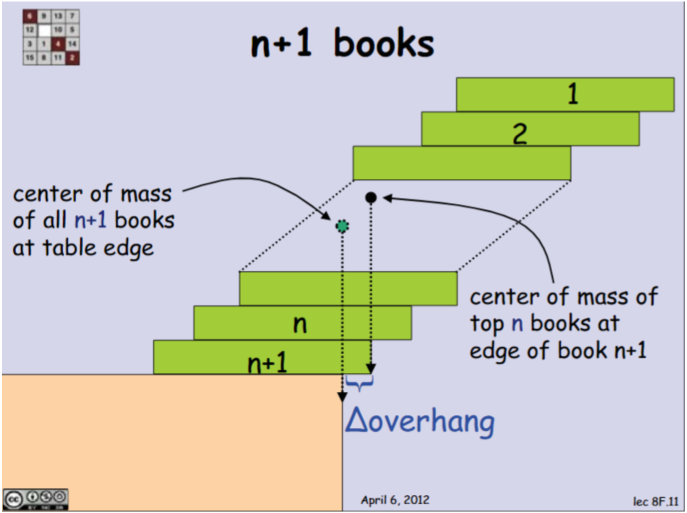

Figure 4: Albert R. Meyer, MIT 6.042J

Balancing Torque Equation

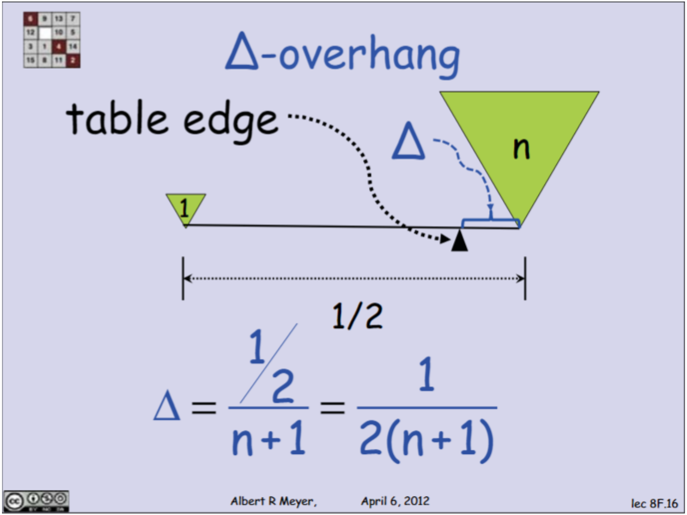

Figure 5: Albert R. Meyer, MIT 6.042J

Book Stacking Formula

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We can explicitly write the function as $B_n = \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) = H_n/2$.

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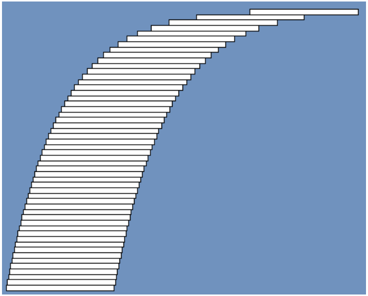

Figure 6: The overhang for 52 cards is 2.269 times the width of one card.

Outline

Relevant Links

- Numberphile: The mystery of 0.577
- Ed Sandifer: How Euler Did It
- Adrien Dudek: Divergence of $\sum_p \frac{1}{p}$
- Charles Jordan: Thirty Card Mysteries
- Albert R. Meyer: Book Stacking Video (MIT 6.042J)
- Brian Brushwood: The Leaning Tower of Cardsa