Divisibility Solutions

Justin Stevens

Problem 1. (AIME) Find the sum of all positive two-digit integers that are divisible by each of their digits.

Solution. Let the two digit number be $N = 10a+b$. We see that we must have $a \mid 10a+b$ and $b \mid 10a+b$. The first condition gives $a \mid b$ and the second gives $b \mid 10a$. Therefore, we must either have $b = a, b = 2a$, or $b = 5a$. These give the resulting values of $N = 11a, N = 12a$, and $N = 15a$.

- If $N = 11a$, then $N = 11, 22, 33, 44, 55, 66, 77, 88, 99$ which has a sum of $11\left(\frac{9\cdot 10}{2}\right) = 11 \cdot 45 = 495$.
- If $N = 12a$, then $N = 12, 24, 36, 48$ for a sum of 120.
- If $N = 15a$, then $N = 15$ for a sum of 15.

Hence, the smallest sum is $495 + 120 + 15 = \boxed{630}$. □

Problem 2. (HMMT) For what single digit N does 91 divide the 9-digit number $12345N789$.

Solution. Observe that $1001 = 91 \cdot 11$. Therefore, we have $1000 \equiv -1 \pmod{91}$. Notice that

$$123450789 = 123 \cdot 10^6 + 450 \cdot 10^3 + 789 \equiv 32 + (-5) \cdot -1 + 61 = 98 \equiv 7 \pmod{91}.$$

Therefore, we see that $12345N789 = 7 + 1000N \equiv 7 - N \equiv 0 \pmod{91} \implies N = \boxed{7}$. □

Problem 3. (HMMT) What is the smallest 5-digit palindrome that is a multiple of 99?

Solution. Observe that a 5-digit palindrome is of the form

$$XYZYX = 10000X + 1000Y + 100Z + 10Y + X = 10001X + 1010Y + 100Z.$$

Notice that $99 \mid 9999$ and $99 \mid 990$, therefore, we see that $10001 \equiv 2 \pmod{99}$ and $1010 \equiv 20 \pmod{99}$. Hence, the entire expression is

$$10001X + 1010Y + 100Z \equiv 2X + 20Y + Z \equiv 0 \pmod{99}.$$

Since X, Y, and Z are all digits, we have $0 \leq X, Y, Z \leq 9$. Hence, $2X + 20Y + Z = 99$ or $2X + 20Y + Z = 198$.

- In the first case, the smallest palindrome we obtain is when $Y = 4$ giving $2X + Z = 19$. We must have $Z = 9 \implies X = 5$, giving the palindrome 54945.

- In the second case, the smallest palindrome we obtain is when $Y = 9$ giving $2x+Z = 18$. We must have $Z = 8 \implies X = 5$, giving the palindrome 59895.

The smaller of the two cases is $\boxed{54945}$. □

Problem 4. (AMC) When we divide the numbers $1059, 1417$, and 2312 by an integer $d > 1$, the remainder is the integer r. Find d and r.

Solution. By the division algorithm, there exists three integers quotients q_1, q_2, and q_3 such that

$$\begin{aligned} 1059 &= dq_1 + r \\ 1417 &= dq_2 + r \\ 2312 &= dq_3 + r. \end{aligned}$$

Subtracting the equations in pairs gives:

$$358 = d\left(q_2 - q_1\right), 895 = d\left(q_3 - q_2\right), 1253 = d\left(q_3 - q_1\right).$$

Hence, d must divide $358 = 2 \cdot 179, 895 = 5 \cdot 179, 1253 = 7 \cdot 179$. We therefore conclude that $d = \boxed{179}$. Now, dividing 1059 by 179 gives $1059 = 179 \cdot 5 + 164$. Therefore, $r = \boxed{164}$. □

Problem 5. Recall the divisibility rule for 7, which claimed that if $7 \mid 10a+x$, then $7 \mid a-2x$.

(i) Prove that if $13 \mid 10a + x$, then $13 \mid a + 4x$.

Solution. Let $N = 10a + x$. Since $13 \mid N$, we must have $13 \mid 4N$. Observe that

$$4N = 40a + 4x = 39a + (a + 4x).$$

Therefore, since $13 \mid 4N$ and $13 \mid 39a$, we must have $13 \mid a + 4x$. □

(ii) State and prove a divisibility rule for 17.

Solution. The divisibility rule for 17 is if $17 \mid 10a + x$, then $17 \mid a - 5x$.
To prove this, observe that if $17 \mid N$, then

$$17 \mid 12N = 120a + 12x = 119a + 17x + (a - 5x).$$

Since $17 \mid 119a + 17x$, we must have $17 \mid a - 5x$.
Note: The key insight in discovering this was that $12 \cdot 10 \equiv 1 \pmod{17}$. □

Problem 6. Use the definition of divisibility to show the following three statements.

(i) Show that if $d \mid n_1$ and $d \mid n_2$, then $d \mid n_1c_1 + n_2c_2$ for integers c_1 and c_2.

Solution. Since d divides n_1, there exists an integer q_1 such that $n_1 = q_1 d$. Similarly, there exists an integer q_2 such that $n_2 = q_2 d$. Therefore,

$$\begin{aligned} n_1 x_1 + n_2 x_2 &= (q_1 d)\, x_1 + (q_2 d)\, x_2 \\ &= d\,(q_1 x_1 + q_2 x_2)\,. \end{aligned}$$

Hence, d divides $n_1 x_1 + n_2 x_2$. In other words, if $d \mid n_1$ and $d \mid n_2$, then d divides all linear combinations of n_1 and n_2. □

(ii) Show that if $d \mid m$ and $m \mid n$, then $d \mid n$.

Solution. Since $d \mid m$, there exists an integer q_1 such that $m = q_1 d$. Since $m \mid n$, there exists an integer q_2 such that $n = q_2 m$. Substituting for n and m, we see that

$$n = q_2 m = q_2\,(q_1 d) = d\,(q_2 q_1)\,.$$

Hence, $d \mid n$. For instance, $7 \mid 49 \mid 98$, therefore, $7 \mid 98$. □

(iii) Show that if $dc \mid nc$, then $d \mid n$.

Solution. Since $dc \mid nc$, this implies there exists an integer q such that

$$nc = q\,(dc)\,.$$

Dividing this equation by c, we arrive at $n = qd$, or alternatively, $d \mid n$. □

Problem 7. Recall from class that all perfect squares are of the form $3k$ or $3k + 1$.

(i) Show that every perfect square is of the form $4k$ or $4k + 1$.

Solution. Let the perfect square be of the form n^2. If n is even, then $n = 2m$ for some integer m. Hence, $n^2 = 4m^2$ which is of the form $4k$. Alternatively, if n is odd, then $n = 2m + 1$ for some integer m. Hence,

$$n^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 4\left(m^2 + m\right) + 1,$$

which is of the form $4k + 1$ where $k = m^2 + m$. □

(ii) Show that every perfect square is of the form $5k$, $5k + 1$, or $5k + 4$.

Solution. We break this into casework based on the remainder when n is divided by m.

(a) If the number is of the form $n = 5m$, then $n^2 = 25m^2 = 5\,(5m^2)$.

(b) If the number is of the form $n = 5m + 1$, then

$$n^2 = 25m^2 + 10m + 1 = 5\left(5m^2 + 2m\right) + 1.$$

(c) If the number is of the form $n = 5m + 2$, then

$$n^2 = 25m^2 + 20m + 4 = 5\left(5m^2 + 4m\right) + 4.$$

(d) If the number is of the form $n = 5m + 3$, then

$$n^2 = 25m^2 + 30m + 9 = 5\left(5m^2 + 6m + 1\right) + 4.$$

(e) If the number is of the form $n = 5m + 4$, then

$$n^2 = 25m^2 + 40m + 16 = 5\left(5m^2 + 8m + 3\right) + 1.$$

Note: For the last two, we could have alternatively considered $n = 5m - 2$ and $n = 5m - 1$. ☐

($\star$) Show that for all integers a and b, $ab(a^2 - b^2)(a^2 + b^2)$ is divisible by 30.

Solution. We break this into three parts: proving that 2 divides the expression, 3 divides the expression, and 5 divides the expression.

- To show that 2 divides the expression is easy: if either a or b is even, then clearly the expression is as well. Alternatively, if both a and b are odd, then $a^2 - b^2$ is even.

- To prove 3, if $3 \mid a$ or $3 \mid b$, then clearly 3 divides the product. If $3 \nmid a$ and $3 \nmid b$, then we have $a^2 \equiv b^2 \equiv 1 \pmod 3$. therefore, $3 \mid a^2 - b^2$.

- To prove 5, if $5 \mid a$ or $5 \mid b$, then 5 divides the product. If $5 \nmid a$ and $5 \nmid b$, then we have four cases: $a^2 \equiv b^2 \equiv 1 \pmod 5$, $a^2 \equiv b^2 \equiv 4 \pmod 5$, or one of them is 1 and the other is 4. In the first two cases, we have $a^2 - b^2 \equiv 0 \pmod 5$. In the third case, we see that $a^2 + b^2 \equiv 1 + 4 \equiv 0 \pmod 5$. In every case, 5 divides the product.

Since we have shown that $2, 3$, and 5 all divide the product, we conclude that the product is divisible by 30. ☐

Problem 8. The nth triangular number is defined as $T_n = 1 + 2 + 3 + \cdots + n$.

(i) Show that $T_n = \frac{n(n+1)}{2}$.

Solution. We note that adding the sum reserved gives

$$\begin{aligned} T_n &= 1 + \quad 2 \quad + \quad 3 \quad + \cdots + n \\ T_n &= n + (n-1) + (n-2) + \cdots + 1. \end{aligned}$$

If we add the two equations columnwise, we see that each pair adds to $n+1$. Therefore:

$$2T_n = n(n+1) \implies T_n = \boldsymbol{\frac{n(n+1)}{2}}.$$

☐

(ii) Show that the quantity $8T_n + 1$ is always a perfect square.

Solution. Observe that

$$8T_n + 1 = 8\left(\frac{n(n+1)}{2}\right) + 1 = 4n^2 + 4n + 1 = (2n+1)^2.$$

$\square$

(iii) Show that the sum of two consecutive triangular numbers is always a perfect square.

Solution. Observe that

$$T_{n-1} + T_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2} = n\left(\frac{n-1+n+1}{2}\right) = n \cdot n = n^2.$$

$\square$

(iv) Show that $9T_n + 1$ is always a triangular number.

Solution. Observe that

$$9T_n + 1 = 9\left(\frac{n(n+1)}{2}\right) + 1 = \frac{9n^2 + 9n + 2}{2} = \frac{(3n+2)(3n+1)}{2} = T_{3n+2}.$$

$\square$

($\star$) Find a formula for the sum of the first n triangular numbers.

Solution. We color the triangular numbers blue. We compute the top row by beginning with the circled 0 and adding the triangular number below it.

$$\begin{array}{ccccccccccccccccc}
\textcircled{0} & & 1 & & 4 & & 10 & & 20 & & 35 & & 56 & & 84 & & 120 \\
 & \color{blue}{1} & & \color{blue}{3} & & \color{blue}{6} & & \color{blue}{10} & & \color{blue}{15} & & \color{blue}{21} & & \color{blue}{28} & & \color{blue}{36} &
\end{array}$$

Continuing the method of finite differences below this gives:

$$\begin{array}{ccccccccccccccccc}
\textcircled{0} & & 1 & & 4 & & 10 & & 20 & & 35 & & 56 & & 84 & & 120 \\
 & \color{blue}{1} & & \color{blue}{3} & & \color{blue}{6} & & \color{blue}{10} & & \color{blue}{15} & & \color{blue}{21} & & \color{blue}{28} & & \color{blue}{36} & \\
 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & &
\end{array}$$

Since the third row is constant, we expect our formula to be cubic. Let $p(n) = an^3 + bn^2 + cn + d$. From the circled number, $p(0) = 0 \implies d = 0$. Plugging in $p(1) = 1, p(2) = 4$, and $p(3) = 10$ gives:

$$\begin{cases} a + b + c = 1 \\ 8a + 4b + 2c = 4 \\ 27a + 9b + 3c = 10. \end{cases}$$

Solving this system gives $(a, b, c) = (\frac{1}{6}, \frac{1}{2}, \frac{1}{3})$. Hence,

$$p(n) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n = \boxed{\frac{n(n+1)(n+2)}{6}}.$$

$\square$