
Intermediate Number Theory

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FOURTH EDITION

Mathematics is the queen of sciences and number theory is the queen of mathematics.

– Carl Friedrich Gauss

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§2.1 Basic Theorems

Definition. For integers n and $d \neq 0$, d divides n , written $d \mid n$, if and only if there exists an integer q such that $n = dq$. This is equivalent to saying n is a multiple of d .

For every integer n , the definition directly implies $1 \mid n$, $n \mid 0$, and $n \mid n$.

Theorem 2.1. Properties of divisibility over the integers include:

- (1) **Transitive:** $d \mid m$ and $m \mid n$ implies $d \mid n$.
- (2) **Linear Combinations:** $d \mid n_1$ and $d \mid n_2$ implies $d \mid n_1\alpha_1 + n_2\alpha_2$ for integers α_1, α_2 .
- (3) **Cancellation:** $dc \mid nc$ implies $d \mid n$.
- (4) **Anti-symmetry:** $d \mid n$ and $n \mid d$ implies $|n| = |d|$.

Proof. We prove (1) and (2), leaving (3) and (4) as exercises.

(1) Since $d \mid m$, there exists $q \in \mathbb{Z}$ with $m = qd$. Similarly, $n = q'm$ for $q' \in \mathbb{Z}$. Hence,

$$n = q'm = q'(qd) = d(q'q) \implies d \mid n.$$

(2) Since $d \mid n_1$, there exists $q_1 \in \mathbb{Z}$ with $n_1 = q_1d$. Similarly, $n_2 = q_2d$ for $q_2 \in \mathbb{Z}$. Hence,

$$\begin{aligned} n_1\alpha_1 + n_2\alpha_2 &= (q_1d)\alpha_1 + (q_2d)\alpha_2 \\ &= d(q_1\alpha_1 + q_2\alpha_2). \end{aligned}$$

Hence, $d \mid n_1\alpha_1 + n_2\alpha_2$. These numbers are called linear combinations of n_1 and n_2 . □

Example 2.1 (AIME). Find the sum of all positive two-digit integers that are divisible by each of their digits.

Solution. Let the number be $N = 10a + b$. The first condition gives $a \mid b$ while the second condition gives $b \mid 10a$, therefore $b = a$, $b = 2a$, or $b = 5a$. Therefore, $N = 11a$, $N = 12a$, or $N = 15a$.

- If $N = 11a$, then we have a sum of $11 \left(\frac{9 \cdot 10}{2}\right) = 11 \cdot 45 = 495$.
- If $N = 12a$, then $N = 12, 24, 36, 48$ for a sum of 120.
- If $N = 15a$, then $N = 15$ for a sum of 15.

Hence, the total is $495 + 120 + 15 = \boxed{630}$. □

Example 2.2. Prove that if $d \mid n$ and $(d + 1) \mid n$, then $(d^2 + d) \mid n$.

Solution. From the given conditions, there exists integers r and s such that

$$n = dr = (d + 1)s.$$

Taking the reciprocal of these equations and subtracting gives

$$\frac{1}{d} - \frac{1}{d + 1} = \frac{r}{n} - \frac{s}{n} \implies \frac{1}{d(d + 1)} = \frac{r - s}{n}.$$

Therefore, we conclude that $(d^2 + d) \mid n$. □

Example 2.3. Prove that if $d \mid n$, then $F_d \mid F_n$.

Proof. Note $F_d \mid F_{d \cdot 1}$. Let $s = qd$ and $t = d$ in the identity $F_{s+t} = F_{s+1}F_t + F_sF_{t-1}$:

$$F_{qd+d} = F_{qd+1}F_d + F_{qd}F_{d-1}.$$

If F_d divides F_{qd} , it also divides their linear combination, $F_{(q+1)d}$. The result follows by induction. □

Example 2.4. Prove that the product of n consecutive integers is divisible by $n!$.

Solution. For positive m , we consider the Binomial coefficient

$$\binom{m+n}{n} = \frac{(m+n)!}{m!n!} = \frac{(m+n)(m+n-1)\cdots(m+1)}{n!}.$$

Since the left-hand side is an integer, $n!$ divides the product of n consecutive positive integers. If any of the consecutive integers is 0, then the product is 0, and we are done. Otherwise, if the n consecutive integers are all negative, we multiply by $(-1)^n$ to reduce to the positive case. □

Exercises

2.1.1. Find all three-digit odd numbers that are divisible by each of their distinct digits.

2.1.2. Prove the cancellation and anti-symmetry properties of divisibility.

2.1.3. Prove that $n^5 - 5n^3 + 4n$ is always divisible by 120 for all integers n .

2.1.4. Prove that if d divides every integer in the sequence n_1, n_2, \dots, n_k , then

$$d \mid n_1\alpha_1 + n_2\alpha_2 + \cdots + n_k\alpha_k, \quad \alpha_j \in \mathbb{Z}.$$

§2.2 Primes and Algebraic Identities

Definition. A prime is a number $p > 1$ whose only positive divisors are 1 and itself.

Definition. A composite number n can be written as $n = ab$ for $a > 1$ and $b > 1$.

Since divisors of composites come in pairs, if $d \nmid n$ for $2 \leq d \leq \lfloor \sqrt{n} \rfloor$, then n is prime.

Sieve (Eratosthenes). Write the numbers 1 to N in a grid. For all primes $p \leq \sqrt{N}$, cross out the multiples $2p, 3p, 4p, \dots$. The numbers that remain are the primes less than N .

Example 2.5. Find all primes less than or equal to 100.

Solution. We show the completed grid using the Sieve of Eratosthenes:

	②	③	4	⑤	6	⑦	8	9	10
⑪	12	⑬	14	15	16	⑰	18	⑲	20
21	22	⑳	24	25	26	27	28	㉑	30
⑳	32	33	34	35	36	㉓	38	39	40
④①	42	④③	44	45	46	④⑦	48	49	50
51	52	⑤③	54	55	56	57	58	⑤⑨	60
⑥①	62	63	64	65	66	⑥⑦	68	69	70
⑦①	72	⑦③	74	75	76	77	78	⑦⑨	80
81	82	⑧③	84	85	86	87	88	⑧⑨	90
91	92	93	94	95	96	⑨⑦	98	99	100

Example 2.6 (PUMaC). If p, q , and r are primes such that $pqr = 7(p + q + r)$, find $p + q + r$.

Solution. We see that at least one of the primes must be 7, say p . Simplifying using SFFT,

$$qr = q + r + 7 \implies (q - 1)(r - 1) = 8.$$

Therefore, $(q, r) = (5, 3)$ or $(q, r) = (3, 5)$. Hence, $p + q + r = 3 + 5 + 7 = \boxed{15}$. \square

Example 2.7. Find all primes p such that $16p + 1$ is a perfect cube.

Solution. Let $16p + 1 = n^3$ for some integer n . Using difference of cubes,

$$16p = n^3 - 1 = (n - 1)(n^2 + n + 1).$$

Since $n^2 + n + 1$ is odd, we must have $16 \mid n - 1$, therefore $n - 1 = 16k$ for some integer k :

$$p = k(n^2 + n + 1).$$

Since $n^2 + n + 1 > 1$, we must have $k = 1$, so $n = 17$. We then find $p = 17^2 + 17 + 1 = \boxed{307}$. \square

Example 2.8. Find all integers a and b such that $a^4 + 4b^4$ is prime.

Solution. Complete the square by adding $4a^2b^2$ to both sides: $(a^2 + 2b^2)^2 = a^4 + 4a^2b^2 + 4b^4$. Rearranging this equation and using difference of squares:

$$\begin{aligned} a^4 + 4b^4 &= (a^2 + 2b^2)^2 - (2ab)^2 \\ &= (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2) \\ &= [(a - b)^2 + b^2][(a + b)^2 + b^2]. \end{aligned}$$

If $a^4 + 4b^4$ is prime, then one factor must be 1 and the other a prime, so $|a| = |b| = 1$. □

The factorization used in the above problem is known as the **Sophie Germain Identity**.

Example 2.9 (MPfG). Find the unique five-digit prime divisor of 104060465.

Solution. We see $101^4 = 104060401$. Therefore, using the Sophie-Germain identity,

$$\begin{aligned} 104060465 &= 101^4 + 64 = 101^4 + 4 \cdot 2^4 \\ &= (101^2 - 2 \cdot 101 \cdot 2 + 2 \cdot 2^2)(101^2 + 2 \cdot 101 \cdot 2 + 2 \cdot 2^2) \\ &= 9805 \cdot 10613. \end{aligned}$$

The answer is hence 10613. □

Example 2.10. (a) Prove $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})$ for natural n .

(b) Prove $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots + y^{n-1})$ for odd n .

Proof. (a) We use the geometric series formula $\sum_{k=0}^{n-1} r^k = \frac{r^n - 1}{r - 1}$ with $r = \frac{x}{y}$:

$$1 + \frac{x}{y} + \cdots + \frac{x^{n-2}}{y^{n-2}} + \frac{x^{n-1}}{y^{n-1}} = \frac{x^n/y^n - 1}{x/y - 1}$$

Multiplying both sides by y^{n-1} :

$$y^{n-1} + xy^{n-2} + \cdots + x^{n-2}y + x^{n-1} = \frac{x^n - y^n}{x - y}.$$

Multiplying by $x - y$ gives the difference of n th power factorization.

(b) Substitute $-y$ for y in the above factorization when n is odd. □

For $n = 3$, this is the familiar sum and difference of cubes factorization.

Example 2.11 (UMich). Find a prime factor $p > 250000$ of 1002004008016032.

Solution. If $x = 1000$ and $y = 2$, then the given number is $N = x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$. Using the factorization for $n = 6$ along with sum and difference of cubes,

$$\begin{aligned} N &= \frac{1000^6 - 2^6}{1000 - 2} \\ &= 2^5 \cdot \frac{(500^3 - 1)(500^3 + 1)}{500 - 1} \\ &= 2^5 \cdot \frac{(500 - 1)(500^2 + 500 + 1)(500 + 1)(500^2 - 500 + 1)}{500 - 1}. \end{aligned}$$

The only prime factor of this number that satisfies $p > 250000$ is $500^2 + 500 + 1 = \boxed{250501}$. \square

Example 2.12. Prove that if $2^n - 1$ is prime for $n \geq 2$, then n must be prime.

Proof by Contradiction. If n is composite, let $n = cd$ for $c > 1$ and $d > 1$, so

$$\begin{aligned} 2^{cd} - 1 &= (2^c - 1)(2^{c(d-1)} + \dots + 2^{2c} + 2^c + 1) \\ &= (2^d - 1)(2^{d(c-1)} + \dots + 2^{2d} + 2^d + 1). \end{aligned}$$

However, then $2^n - 1$ is also composite, contradiction. Thus, n must be prime. \square

Definition. Primes of the form $M_p = 2^p - 1$ are called **Mersenne primes**.

There are currently 49 known Mersenne primes. As of this writing, the largest known prime is a Mersenne prime, $2^{74,207,281} - 1$. This number has an astonishing 22,338,618 digits!

Example 2.13. Find a factor of $2^{33} - 2^{19} - 2^{17} - 1$ that lies between 1000 and 5000.

Solution. Observe that for $a = 2^{11}$, $b = -2^6$, and $c = -1$,

$$2^{33} - 2^{19} - 2^{17} - 1 = 2^{33} - 2^{18} - 1 - 3 \cdot 2^{17} = a^3 + b^3 + c^3 - 3abc.$$

Since $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$, the desired divisor is

$$a + b + c = 2^{11} - 2^6 - 1 = 2048 - 64 - 1 = \boxed{1983}. \quad \square$$

Exercises

2.2.1. (Mandelbrot) Jayne writes the integers from 1 to 2000 on a piece of paper. She erases all the multiples of 3, then all the multiples of 5, and so on, erasing all the multiples of each odd prime. How many numbers are left when she finishes?

2.2.2. (ARML) Find all primes p such that $p^{1994} + p^{1995}$ is a perfect square.

2.2.3. (ARML) Find the largest divisor of 1001001001 less than 10000.

2.2.4. Prove that if $d \mid n$, then for positive integers x and y , $x^d - y^d \mid x^n - y^n$.

§2.3 Division Algorithm

Algorithm 2.2 (Division Algorithm). For $n \in \mathbb{Z}$ and $d \in \mathbb{Z}^+$, there exists q and r such that

$$n = dq + r, \quad 0 \leq r < d$$

Proof. For proving the existence of the quotient and remainder, consider the progression:

$$\{\dots, n - 3d, n - 2d, n - d, n, n + d, n + 2d, n + 3d, \dots\}.$$

Let the smallest non-negative member of this arithmetic progression be $r_{\min} = n - dq$. Assume for the sake of contradiction that $r_{\min} \geq d$. Then, we simply subtract d from the value of r to produce $r' = n - d(q + 1) \geq 0$. Since $r' < r_{\min}$, we have found a smaller non-negative member of the progression than r , contradiction. Therefore, $0 \leq r_{\min} < d$.

The second part of this proof is to show that the quotient and remainder are unique. Assume for the sake of contradiction that n can be represented in two ways:

$$n = dq_1 + r_1 = dq_2 + r_2 \implies d(q_1 - q_2) = r_2 - r_1.$$

This implies that $r_2 - r_1$ is a multiple of d . However, $-d < r_2 - r_1 < d$ since $0 \leq r_1, r_2 < d$. Therefore, if $r_2 - r_1$ is a multiple of d , we must have $r_2 - r_1 = 0 \implies r_2 = r_1$ and $q_2 = q_1$. \square

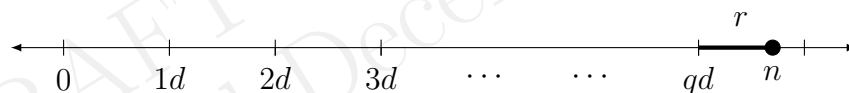


Figure 2.1: If $d \nmid n$, then n will always lie between two ticks on the number line.

Example 2.14 (AHSME). Let r be the remainder when each of the numbers 1059, 1417, and 2312 are divided by an integer $d > 1$. Find d and r .

Solution. By the division algorithm, there exists three integer quotients q_1, q_2 , and q_3 with

$$1059 = dq_1 + r, \quad 1417 = dq_2 + r, \quad 2312 = dq_3 + r.$$

We subtract the equations in pairs:

$$358 = d(q_2 - q_1), \quad 895 = d(q_3 - q_2), \quad 1253 = d(q_3 - q_1).$$

Hence, d must divide $358 = 2 \cdot 179$, $895 = 5 \cdot 179$, and $1253 = 7 \cdot 179$, so $d = \boxed{179}$. Dividing 1059 by 179 shows $1059 = 179 \cdot 5 + 164$, therefore the remainder is $r = \boxed{164}$. \square

Another application of the division algorithm is different forms of numbers. For integer k , define A to be the set of numbers of the form $3k$, B to be $3k + 1$, and C to be $3k + 2$. Notice that these sets make up the entirety of the integers, therefore $A \cup B \cup C = \mathbb{Z}$.

Example 2.15. Prove that every perfect square is of the form $3k$ or $3k + 1$.

Solution. We break this into three cases based on the remainder when we divide n by 3. If $n = 3m$, then $n^2 = (3m)^2 = 9m^2 = 3(3m^2)$, which is of the form $3k$. Otherwise,

$$\begin{aligned} n^2 &= (3m + 1)^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1 \\ n^2 &= (3m + 2)^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1. \end{aligned}$$

In both cases, we conclude that n^2 is of the form $3k + 1$. □

Example 2.16. Prove that $n^9 - 6n^7 + 9n^5 - 4n^3$ is divisible by 8640 for all integers $n \geq 1$.

Proof. We begin by factorizing the polynomial:

$$\begin{aligned} n^9 - 6n^7 + 9n^5 - 4n^3 &= n^3 (n^6 - 6n^4 + 9n^2 - 4) \\ &= n^3 (n^2 - 1) (n^4 - 5n^2 + 4) \\ &= n^3 (n^2 - 1) (n^2 - 1) (n^2 - 4) \\ &= (n - 2) (n - 1)^2 n^3 (n + 1)^2 (n + 2). \end{aligned}$$

Since $8640 = 2^6 \cdot 3^3 \cdot 5^1$, we break the problem into the prime divisors:

- 5^1 : We have 5 consecutive integers in the product, so the product is divisible by 5.
- 3^3 : If $n = 3k$, then n^3 has three factors of 3. If $n = 3k + 1$, then $(n - 1)^2 (n + 2)$ has three factors of 3. Finally, if $n = 3k + 2$, then $(n - 2) (n + 1)^2$ has three factors of 3.
- 2^6 : If $n = 4k$, then $(n - 2) n^3 (n + 2)$ has eight factors of 3. If $n = 4k + 2$, then it has seven factors of 3. If $n = 4k + 1$ or $n = 4k + 3$, then $(n - 1)^2 (n + 1)^2$ has six factors of 3.

For every prime, we've proven the polynomial has enough factors to be divisible by 8640. □

Example 2.17 (TAG₅). Let there be n coins and two players A and B , with A making the first move. Each player may in turn take-away either 1, 2, 3, 4, or 5 coins from the pile. The winner is the player who removes the final coin(s). Determine the winning strategy for general n .

Solution. If $6 \mid n$ and player A takes s coins, player B takes $6 - s$ coins to reduce the pile by 6. Player B can repeat this strategy until there are $1 \leq r \leq 5$ coins remaining to win. Else if $6 \nmid n$, player A can take r coins to leave $n' = n - r$ coins where $6 \mid n'$ to win using the strategy above. □

Exercises

2.3.1. Find the greatest three digit integer that leaves a remainder of 7 upon division by 13.

2.3.2. (AHSME) Let S be a subset of $\{1, 2, 3, \dots, 50\}$ such that no pair of distinct elements in S has a sum divisible by 7. What is the maximum number of elements in S ?

2.3.3. Euclid divides 2017 by every positive integer up to 1000 using the division algorithm and records the remainder. What is the largest remainder Euclid writes down?

2.3.4. Prove the product of two numbers of the form $4k + 3$ is of the form $4k + 1$.

§2.4 Modular Arithmetic

Concept (Modular Arithmetic). We define the quotient group \mathbb{Z}_m by:

- The mod m residues are the remainders upon division by m , $0, 1, 2, \dots, m - 1$.
- Integers a and b are said to be congruent mod m if they have the same residue:

$$\begin{aligned} a &= q_1m + r \\ b &= q_2m + r, \end{aligned}$$

where q_1, q_2 , and r are integers with $0 \leq r < m$.

- This is written as $a \equiv b \pmod{m}$ and is equivalent to $m \mid a - b$.

We see that if $a \equiv b \pmod{m}$, then $a - b = mq$ for some integer q , implying $a = b + mq$.

Theorem 2.3. If $a_1 \equiv a_2 \pmod{m}$ and $b_1 \equiv b_2 \pmod{m}$, then

$$a_1 + b_1 \equiv a_2 + b_2 \pmod{m}, \quad a_1b_1 \equiv a_2b_2 \pmod{m}.$$

Proof. By definition, there exists integers q_a and q_b such that $a_1 = a_2 + mq_a$ and $b_1 = b_2 + mq_b$. Adding these equations gives

$$a_1 + b_1 = a_2 + mq_a + b_2 + mq_b \equiv a_2 + b_2 \pmod{m}.$$

Multiplying these equations gives

$$a_1b_1 \equiv (a_2 + mq_a)(b_2 + mq_b) = a_2b_2 + mq_ab_2 + mq_ba_2 + m^2q_aq_b \equiv a_2b_2 \pmod{m}. \quad \square$$

Example 2.18. Compute the units digit of $1! + 2! + 3! + 4! + 5! + \dots + 100!$.

Solution. Since $5! = 120 \equiv 0 \pmod{10}$, every term after that cancels, so the units digit is

$$1! + 2! + 3! + 4! + 5! \dots + 100! \equiv 1 + 2 + 6 + 4 + 0 + \dots + 0 \equiv \mathbf{3} \pmod{10}. \quad \square$$

Example 2.19 (AIME). Find the remainder when $9 \times 99 \times \dots \times \underbrace{99 \dots 9}_{999 \text{ 9's}}$ is divided by 1000.

Solution. After the second term, every number is $-1 \pmod{1000}$, so the product is

$$9 \times 99 \times \underbrace{999 \dots 9}_{999 \text{ 9's}} \equiv 9 \cdot 99 \cdot (-1)^{997} \equiv -891 \equiv \mathbf{109} \pmod{1000}. \quad \square$$

Theorem 2.4 (Modulo Exponentiation). If $a_1 \equiv a_2 \pmod{m}$, then $a_1^n \equiv a_2^n \pmod{m}$.

We can prove this by repeatedly multiplying together the congruence $a_1 \equiv a_2 \pmod{m}$.

Example 2.20. Compute the remainder when 6^{203} is divided by 37.

Solution. Observe that $6^2 \equiv 36 \equiv -1 \pmod{37}$. Therefore,

$$6^{203} \equiv (6^2)^{101} \cdot 6^1 \equiv (-1)^{101} \cdot 6^1 \equiv \boxed{31} \pmod{37}. \quad \square$$

Example 2.21. Compute the units digit of $3^{7^{11}}$.

Solution. Observe the following relations for nonnegative integer n :

$$n \equiv 0 \pmod{4} \implies 3^n \equiv 1 \pmod{10}$$

$$n \equiv 1 \pmod{4} \implies 3^n \equiv 3 \pmod{10}$$

$$n \equiv 2 \pmod{4} \implies 3^n \equiv 9 \pmod{10}$$

$$n \equiv 3 \pmod{4} \implies 3^n \equiv 7 \pmod{10}.$$

We therefore wish to find the remainder when 7^{11} is divided by 4:

$$7^{11} \equiv (-1)^{11} \equiv -1 \equiv 3 \pmod{4} \implies 3^{7^{11}} \equiv 3^3 \equiv \mathbf{7} \pmod{10}. \quad \square$$

Example 2.22 (ARML). Find a divisor between 2000 and 3000 of $85^9 - 21^9 + 6^9$.

Solution. Let $N = 85^9 - 21^9 + 6^9$. Since $85 \equiv 21 \pmod{64}$, we see that $85^9 \equiv 21^9 \pmod{64}$. Furthermore, 6^9 is divisible by 64, so N is divisible by 64. Similarly $21 \equiv 6 \pmod{5}$, so

$$N \equiv 0 - 6^9 + 6^9 \equiv 0 \pmod{5}.$$

Finally, since $85 \equiv 1 \pmod{7}$ and $6 \equiv -1 \pmod{7}$, we see that

$$N \equiv 1^9 - 0 + (-1)^9 \equiv 0 \pmod{7}.$$

Therefore, the desired divisor between 2000 and 3000 is $2^6 \cdot 5^1 \cdot 7^1 = \mathbf{2240}$. □

Example 2.23 (AMC 10). Compute the last three digits of 2011^{2011} .

Solution. Since $2011 \equiv 11 \pmod{1000}$, we work mod 1000 and use the Binomial Theorem:

$$\begin{aligned} 2011^{2011} &\equiv (10 + 1)^{2011} \\ &\equiv 10^{2011} + \cdots + \binom{2011}{2} 10^2 + \binom{2011}{1} 10^1 + \binom{2011}{0} \\ &\equiv \frac{2011 \cdot 2010}{2} 10^2 + 2011 \cdot 10^1 + 1 \\ &\equiv 11 \cdot 5 \cdot 100 + 110 + 1 \\ &\equiv \boxed{661} \pmod{1000}. \quad \square \end{aligned}$$

Example 2.24. Find the largest factor of 2 in the prime factorization of $\lfloor \frac{14^{30}}{7^{15} - 1} \rfloor$.

Solution. We begin by computing $14^{30} \pmod{7^{15} - 1}$. Since $7^{15} \equiv 1 \pmod{7^{15} - 1}$, we have

$$14^{30} = 2^{30}7^{30} = 2^{30}(7^{15})^2 \equiv 2^{30} \pmod{7^{15} - 1}.$$

Furthermore, $2^{30} = 4^{15} < 7^{15} - 1$, therefore the greatest integer function is

$$\lfloor \frac{14^{30}}{7^{15} - 1} \rfloor = \frac{14^{30} - 2^{30}}{7^{15} - 1}.$$

We can now factor and use difference of squares:

$$\frac{14^{30} - 2^{30}}{7^{15} - 1} = \frac{2^{30}(7^{30} - 1)}{7^{15} - 1} = 2^{30}(7^{15} + 1).$$

Notice $7^{15} + 1 \equiv (-1)^{15} + 1 \equiv 0 \pmod{8}$. However, since $7^2 = 49 \equiv 1 \pmod{16}$,

$$7^{15} + 1 \equiv (7^2)^7 7^1 + 1 \equiv 8 \pmod{16}.$$

Hence, there are 3 factors of 2 in $7^{15} + 1$. Therefore, the answer is $30 + 3 = \boxed{33}$. \square

Example 2.25. A repunit is a number consisting only of the digit 1, such as 111 and 11111. Let n be a number relatively prime to 10. Prove that there is a repunit divisible by n .

Proof by Contradiction. Let a repunit with j digits be denoted a_j . Assume that there are no repunits divisible by n , hence there are $n - 1$ possible remainders when we divide a_j by n . Considering an infinite number of repunits by the pigeonhole principle, two must have the same remainder upon division by n , say a_i and a_j for $j > i$. However, then

$$a_j - a_i = \underbrace{111 \cdots 111}_{j \text{ 1's}} - \underbrace{11 \cdots 11}_{i \text{ 1's}} = 10^i \cdot \underbrace{11 \cdots 11}_{j-i \text{ 1's}} = 10^i a_{j-i}.$$

Since a_j and a_i have the same remainder, n divides their difference. Furthermore, n is relatively prime to 10, so $n \mid a_{j-i}$, contradiction. Therefore, we can always find a repunit divisible by n . \square

Exercises

2.4.1. Prove that if $a_1 \equiv a_2 \pmod{m}$ and $d \mid m$, then $a_1 \equiv a_2 \pmod{d}$.

2.4.2. (a) Find the remainder when 5^{1337} is divided by 31.

(b) Find the remainder when 2^{2001} is divided by $2^7 - 1$. (*Source: HMMT*)

2.4.3. Prove that for all natural numbers n , $2^{4n} + 10n - 1$ is divisible by 25.

2.4.4. Prove that if $a \equiv b \pmod{n}$, then $a^n \equiv b^n \pmod{n^2}$.

§2.5 Base Numbers

Definition. When writing numbers using the first b whole numbers (i.e. $0, 1, 2, \dots, b-1$), we are using base b . We can think of base conversions as different ways of grouping numbers.

In general, we write a number n in base b as

$$n = (d_k d_{k-1} \cdots d_2 d_1 d_0)_b = d_k b^k + d_{k-1} b^{k-1} + \cdots + d_2 b^2 + d_1 b + d_0, \quad 0 \leq d_i < b.$$

For example, to convert 2017 to base 8, we repeatedly divide using the division algorithm:

$$2017 = 8 \cdot 252 + \mathbf{1}, \quad 252 = 8 \cdot 31 + \mathbf{4}, \quad 31 = 8 \cdot 3 + \mathbf{7}, \quad 3 = 8 \cdot 0 + \mathbf{3}.$$

Therefore, $2017 = 8 \cdot 252 + 1 = 8(8 \cdot 31 + 4) + 1 = 8(8(8 \cdot 3 + 7) + 4) + 1 = 3741_8$.

Algorithm 2.5 (Base Conversion). To express a decimal number n in base b , we repeatedly divide n by b using the division algorithm, stopping when the quotient is 0:

$$\begin{aligned} n &= b \cdot q_0 + r_0 \\ q_0 &= b \cdot q_1 + r_1 \\ q_1 &= b \cdot q_2 + r_2 \\ &\dots \\ q_{k-2} &= b \cdot q_{k-1} + r_{k-1} \\ q_{k-1} &= b \cdot 0 + r_k. \end{aligned}$$

Then n is the result of appending the remainders in reverse order, $(r_k r_{k-1} r_{k-2} \cdots r_2 r_1 r_0)_b$.

Another algorithm employs a greedy strategy by continually subtracting the largest power of b that goes into n . For bases that are greater than 10, we use letters to represent digits. For example, in base 16, we use A, B, C, D, E, F to represent values ten to fifteen.

Example 2.26. Convert 11110011_2 to base 16 and 25681_9 to base 3.

Solution. For the first part, we group the bits in fours to see

$$11110011_2 = (2^7 + 2^6 + 2^5 + 2^4) + (2^1 + 2^0) = 15 \cdot 2^4 + 3 = \boxed{F3}_{16}.$$

For the second part, since $9 = 3^2$,

$$\begin{aligned} 25681_9 &= 2 \cdot 9^4 + 5 \cdot 9^3 + 6 \cdot 9^2 + 8 \cdot 9^1 + 1 \cdot 9^0 \\ &= 2 \cdot 3^8 + 5 \cdot 3^6 + 6 \cdot 3^4 + 8 \cdot 3^2 + 1 \cdot 3^0. \end{aligned}$$

Converting 5, 6, and 8 to base 3 gives:

$$\begin{aligned} &= 2 \cdot 3^8 + (3 + 2) \cdot 3^6 + (3 \cdot 2) \cdot 3^4 + (3 \cdot 2 + 2) \cdot 3^2 + 1 \cdot 3^0 \\ &= 2 \cdot 3^8 + 1 \cdot 3^7 + 2 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^3 + 2 \cdot 3^2 + 1 \cdot 3^0 \\ &= \boxed{212202201}_3. \end{aligned}$$

We could also directly convert each digit to base 3: $2 = 2_3, 5 = 12_3, 6 = 20_3, 8 = 22_3, 1 = 01_3$. \square

Example 2.27. Convert 100_{b+1} , 1000_{b+1} , and 10000_{b+1} to base b .

Solution. Using the Binomial Theorem,

$$\begin{aligned} 100_{b+1} &= (b+1)^2 = b^2 + 2b + 1 = \mathbf{121}_b \\ 1000_{b+1} &= (b+1)^3 = b^3 + 3b^2 + 3b + 1 = \mathbf{1331}_b \\ 10000_{b+1} &= (b+1)^4 = b^4 + 4b^3 + 6b^2 + 4b + 1 = \mathbf{14641}_b. \end{aligned} \quad \square$$

Example 2.28 (AIME). Call N a *7-10-double* if the digits of the base-7 representation of N form a base-10 number that is twice N . For example, $51 = 102_7$. What is the largest *7-10-double*?

Solution. Let the base 7 representation of the *7-10-double* be $N = (a_k a_{k-1} \cdots a_2 a_1 a_0)_7$. Hence,

$$(a_k a_{k-1} \cdots a_2 a_1 a_0)_{10} = 2(a_k a_{k-1} \cdots a_2 a_1 a_0)_7.$$

Observe that for $k \geq 3$, we have $10^k \gg 2 \cdot 7^k$, so we try $k = 2$:

$$100a_2 + 10a_1 + a_0 = 98a_2 + 14a_1 + 2a_0 \implies 2a_2 = 4a_1 + a_0.$$

To maximize N , we let $a_2 = 6$, giving $a_1 = 3$ and $a_0 = 0$. Therefore, $N = 630_7 = \mathbf{315}$. □

Example 2.29 (ARML). Let $N_b = 1_b + 2_b + \cdots + 100_b$ for an integer $b > 2$. Compute the number of values of b for which the sum of the squares of the digits of N_b is at most 512.

Solution. We convert every term in the sum for N_b to decimal:

$$N_b = 1 + 2 + \cdots + b^2 = \frac{b^2(b^2 + 1)}{2} = \frac{b^4}{2} + \frac{b^2}{2}.$$

If b is even, $N_b = \frac{b}{2}b^3 + \frac{b}{2}b = (\frac{b}{2}0\frac{b}{2}0)_b$. Since the sum of the squares of the digits is at most 512,

$$\left(\frac{b}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = \frac{b^2}{2} \leq 512 \implies b^2 \leq 1024 \implies 4 \leq b \leq 32.$$

There are **15** even values of b in this range. If b is odd, then $N_b = \frac{b-1}{2}b^3 + \frac{b+1}{2}b^2 = (\frac{b-1}{2}\frac{b+1}{2}00)_b$, so

$$\left(\frac{b-1}{2}\right)^2 + \left(\frac{b+1}{2}\right)^2 \leq 512 \implies \frac{b^2 + 1}{2} \leq 512 \implies 3 \leq b \leq 31.$$

There are **15** odd values of b in this range. In conclusion, there are $15 + 15 = \boxed{30}$ values for b . □

Example 2.30 (AIME). The increasing sequence $1, 3, 4, 9, 10, 12, 13, \dots$ consists of all those positive integers which are exponent powers of 3 or sums of distinct powers of 3. Find the 100th term of this sequence.

Solution. We create a table of the powers of 3 used for the first several terms:

k	k th term	3^2	3^1	3^0
1	1	0	0	1
2	3	0	1	0
3	4	0	1	1
4	9	1	0	0
5	10	1	0	1
6	12	1	1	0
7	13	1	1	1

We see $100 = 64 + 32 + 4 = 1100100_2$. Since the 1's and 0's actually correspond to powers of 3, the 100th term of the sequence is

$$3^6 + 3^5 + 3^2 = \mathbf{981}.$$

□

Fractional bases are very similar to normal bases, except they include negative exponents.

Example 2.31. Convert the decimal number $7/16$ to base 6.

Solution. Let the base 6 representation of the decimal number $7/16$ be

$$\frac{7}{16} = \frac{a_1}{6} + \frac{a_2}{6^2} + \frac{a_3}{6^3} + \frac{a_4}{6^4} + \cdots$$

We multiply this equation by 6 and write the left hand side as a mixed number:

$$\frac{42}{16} = 2 + \frac{10}{16} = a_1 + \frac{a_2}{6} + \frac{a_3}{6^2} + \frac{a_4}{6^3} + \cdots$$

Therefore, $a_1 = 2$. We now convert $10/16$ to base 6 using the same method:

$$\frac{60}{16} = 3 + \frac{12}{16} = a_2 + \frac{a_3}{6} + \frac{a_4}{6^2} + \cdots$$

Hence, $a_2 = 3$. Since $12/16 = 4/6 + 3/36$, $a_3 = 4$ and $a_4 = 3$. Hence, $7/16 = \boxed{.2343_6}$. □

Example 2.32 (AIME). A rational number written in base eight is $\underline{ab.cd}$, where all digits are nonzero. The same number in base twelve is $\underline{bb.ba}$. Find the base-ten number \underline{abc} .

Solution. Equating the integer parts, $8a + b = 12b + b \implies a = 3/2b$, so $(a, b) = (3, 2)$ or $(6, 4)$. Similarly, equating the fractional parts,

$$\frac{c}{8} + \frac{d}{64} = \frac{b}{12} + \frac{3/2b}{144} = \frac{3b}{32}.$$

Multiplying by 64 gives $8c + d = 6b$. If $b = 4$, then $(c, d) = (3, 0)$. However the digits are given to be non-zero. If $b = 2$, then $(c, d) = (1, 4)$ and we see $12.34_8 = 22.21_{12}$, so the answer is $\underline{abc} = \mathbf{321}$. □

Example 2.33. Convert $1/(b-1)$ to base b .

Solution. Using the infinite geometric series formula, we see that

$$\frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \cdots = \frac{1/b}{1 - 1/b} = \frac{1}{b-1}.$$

Therefore, $1/(b-1) = \mathbf{1 \cdot b^{-1} + 1 \cdot b^{-2} + 1 \cdot b^{-3} + \cdots = 0.\bar{1}_b}$. □

Example 2.34 (AHSME). Call a positive real number special if it has a decimal representation that consists entirely of digits 0 and 7. For example, $\frac{700}{99} = 7.\overline{07} = 7.070707\cdots$ and 77.007 are special numbers. Find the smallest n such that 1 can be written as a sum of n special numbers?

Solution. Call numbers whose decimal representation consists solely of digits 0 and 1 super-special. We wish to find super-special numbers that sum to $1/7 = 0.\overline{142857}$. One example is

$$.111111, .011111, .010111, .010111, .000111, .000101, .000101, .000100.$$

Assume for the sake of contradiction that we can represent $1/7$ as the sum of less than 8 super-special numbers. However, since the addition won't have any carry over, each digit is simply the number of super-special numbers that have a 1 in the place, so the answer is **8**. \square

Exercises

2.5.1. In balanced ternary, the digits have values $-1, 0$, and 1 . For example, $11 = 3^2 + 3^1 - 3^0$. Find the sum of the exponents used in the balanced ternary representation of 2000.

2.5.2. Prove that 10101 is composite in any number base.

2.5.3. (HMMT) Let S be the set of integers of the form $2^x + 2^y + 2^z$, where x, y, z are pairwise distinct non-negative integers. Determine the 100th smallest element of S .

2.5.4. (AIME) Find the positive integer n such that there exists a single digit d with

$$\frac{n}{810} = 0.d25d25d25\cdots.$$

§2.6 Divisibility Rules

Below is a list of the divisibility rules up until 11.

- 2 - Last digit is even.
- 3 - Sum of the digits is divisible by 3.
- 4 - Number formed by last two digits is divisible by 4.
- 5 - Last digit is either 0 or 5.
- 6 - Divisibility rules for both 2 and 3 hold.
- 7 - Truncate the units digit of N , double that, and subtract it from the rest of the number. Repeat the process if necessary. Check to see if the final number obtained is divisible by 7.
- 8 - Number formed by the last three digits is divisible by 8.
- 9 - Sum of the digits is divisible by 9.
- 10 - The number ends in 0.
- 11 - Alternating sum of digits is divisible by 11.

In this section, we will focus on the rules for 7, 9, and 11.

Example 2.35. Prove that a number is divisible by 9 if and only if the sum of its digits is.

Proof. Notice $10 \equiv 1 \pmod{9}$. Considering the decimal form of a number,

$$\begin{aligned} n &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10^2 a_2 + 10^1 a_1 + a_0 \\ &\equiv 1^k a_k + 1^{k-1} a_{k-1} + \cdots + 1^2 a_2 + 1^1 a_1 + a_0 \\ &\equiv a_k + a_{k-1} + \cdots + a_2 + a_1 + a_0 \pmod{9}. \end{aligned}$$

Hence n is divisible by 9 if and only if the sum of its digits is. \square

Example 2.36 (AHSME). The two-digit integers from 19 to 92 are written consecutively to form the larger number $N = 19202122 \dots 909192$. Find the largest power of 3 that evenly divides N .

Solution. Let $f(n)$ denote the sum of the digits of n . By the divisibility rule for 9,

$$\begin{aligned} f(N) &= f(19) + f(20) + f(21) + \cdots + f(92) \equiv 19 + 20 + 21 + \cdots + 92 \pmod{9} \\ &\equiv 74 \left(\frac{111}{2} \right) \\ &\equiv 3 \cdot 37^2 \equiv 3 \pmod{9}. \end{aligned}$$

Therefore, the highest power of 3 that evenly divides N is $\boxed{3^1}$. \square

Example 2.37. Prove that a number is divisible by 11 iff the alternating sum of its digits is.

Proof. Notice $10 \equiv -1 \pmod{11}$. Considering the decimal form of a number,

$$\begin{aligned} n &= 10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10^2 a_2 + 10^1 a_1 + a_0 \\ &\equiv (-1)^k a_k + (-1)^{k-1} a_{k-1} + \cdots + (-1)^2 a_2 + (-1)^1 a_1 + a_0 \\ &\equiv a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^k a_k \pmod{11}. \end{aligned}$$

Hence n is divisible by 11 if and only if the alternating sum of its digits is. \square

Example 2.38 (PuMaC). If $17! = 355687ab8096000$, find a and b .

Solution. We use both the divisibility rule for 9 and 11:

$$\begin{aligned} a + b + 57 &\equiv 0 \pmod{9} \implies a + b \equiv 6 \pmod{9} \\ a - b + 9 &\equiv 0 \pmod{11} \implies a - b \equiv 2 \pmod{9}. \end{aligned}$$

The solution to this system of congruences is $(a, b) = \boxed{(4, 2)}$. \square

Example 2.39. Find a and b such that the seven-digit number $21358ab$ is divisible by 99.

Solution. Observe that $10^2 \equiv 1 \pmod{99}$, therefore

$$\begin{aligned} 21358ab &= 2 \cdot 100^3 + 13 \cdot 100^2 + 58 \cdot 100 + 10a + b \\ &\equiv 2 + 13 + 58 + 10a + b \\ &\equiv 73 + 10a + b \equiv 0 \pmod{99}. \end{aligned}$$

Therefore, $10a + b = 26$ and $(a, b) = \boxed{(2, 6)}$. □

Example 2.40. Prove a number is divisible by 7 if and only if the result of truncating the units digit of the number and subtracting twice that from the rest of the number is also divisible by 7.

Proof. Let $N = 10a + x$ have units digit x . Multiplying by 5,

$$5N = 50a + 5x \equiv a - 2x \pmod{7}.$$

Therefore, 7 divides $a - 2x$ if and only if 7 divides N . □

Example 2.41. Prove that if a three-digit integer abc is divisible by 37, then its cyclic permutations are also divisible by 37. For example, 481, 814, and 148 are all divisible by 37.

Proof. We are given $100a + 10b + c \equiv 0 \pmod{37}$. Since $10^3 \equiv 1 \pmod{37}$, multiplying by 10 gives

$$1000a + 100b + 10c \equiv 100b + 10c + a \equiv bca \equiv 0 \pmod{37}.$$

Similarly, multiplying our new congruence by 10 gives

$$1000b + 100c + 10a \equiv 100c + 10a + b \equiv cab \equiv 0 \pmod{37}.$$

Therefore, the cyclic permutations of abc are also divisible by 37. □

Example 2.42. (USAMO) Prove that for every positive integer n , there exists an n -digit number divisible by 5^n all of whose digits are odd.

Solution. We use induction. We begin by showing the first 4 base cases: 5, 75, 375, 9375.

Assume that the k digit number $N = a_1a_2a_3 \cdots a_k$ is divisible by 5^k , hence $N = 5^k A$. We show that for one value of $i \in \{1, 3, 5, 7, 9\}$ the number below is divisible by 5^{k+1} :

$$N_i = ia_1a_2a_3 \cdots a_k = i \cdot 10^k + 5^k A = 5^k (i \cdot 2^k + A).$$

Since $\{1, 3, 5, 7, 9\}$ is a complete residue system, there exists an odd i with $i \cdot 2^k + A \equiv 0 \pmod{5}$. For this value of i , N_i is a $k + 1$ -digit number divisible by 5^{k+1} all of whose digits are odd. □

Exercises

2.6.1. A number is selected at random from the set of all five-digit numbers whose sum of digits is equal to 43. What is the probability that this number is divisible by 11?

2.6.2. Prove that a number is divisible by 2^k if and only if the last k digits are.

2.6.3. Modifying the coefficients in the divisibility rule for 7 gives rules for other primes:

- (a) Prove the divisibility rule for 13 replaces $-2x$ with $+4x$.
- (b) Find the prime p whose divisibility rule replaces $-2x$ with $-3x$.

2.6.4. (HMMT) Find the digit N for which 91 divides $12345N789$.

§2.7 Polynomials

A polynomial for a variable x is an expression of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0.$$

The constants $a_n, a_{n-1}, \dots, a_1, a_0$ are the **coefficients**. $a_n \neq 0$ is the leading coefficient and a_0 is the constant coefficient. A polynomial is said to be **monic** if the leading coefficient is 1. n is the **degree** of the polynomial, denoted $\deg(f)$. By convention, the degree of $f(x) = 0$ is undefined.

Theorem 2.6. To divide $n(x)$ by $d(x)$, there exists a unique quotient $q(x)$ and remainder $r(x)$:

$$n(x) = d(x)q(x) + r(x), \quad \deg(r) < \deg(d) \text{ or } r(x) = 0.$$

Proof. To prove the existence, we repeatedly eliminate the leading term of $r(x)$ to decrease its degree until $\deg(r) < \deg(d)$. To prove uniqueness suppose that for quotients q_1 and q_2 ,

$$\begin{aligned} n(x) &= d(x) \cdot q_1(x) + r_1(x) \\ n(x) &= d(x) \cdot q_2(x) + r_2(x), \end{aligned}$$

where $\deg(r_1) < \deg(d)$ (or $r_1 = 0$) and $\deg(r_2) < \deg(d)$ (or $r_2 = 0$). Subtracting and rearranging,

$$d(x)(q_2(x) - q_1(x)) = r_1(x) - r_2(x).$$

However, since $q_1(x)$ and $q_2(x)$ are distinct, we have

$$\deg[d(x)(q_1(x) - q_2(x))] \geq \deg(d(x)).$$

On the other hand, $\deg(r_2(x) - r_1(x)) < \deg(d)$, therefore $r_1(x) = r_2(x)$ and $q_2(x) = q_1(x)$. \square

Theorem 2.7 (Remainder Theorem). The remainder when $f(x)$ is divided by $x - a$ is $f(a)$.

Proof. Since $x - a$ is linear, there exists a constant remainder r upon division:

$$f(x) = (x - a)q(x) + r.$$

Substituting $x = a$ into this expression gives $f(a) = (a - a)q(a) + r \implies f(a) = r$. \square

Example 2.43 (AIME). Find the largest integer n such that $n^3 + 100$ is divisible by $n + 10$.

Solution. The remainder when we divide $p(n) = n^3 + 100$ by $n - (-10)$ is

$$r = p(-10) = -10^3 + 100 = -900.$$

Since $n + 10$ divides the remainder, the largest value of n is when $n + 10 = 900$ or $n = \boxed{890}$. \square

If the remainder when we divide $f(x)$ by $d(x)$ is 0, then we say $d(x)$ is a factor of $f(x)$.

Theorem 2.8 (Factor Theorem). $f(x)$ has a factor of $x - c$ if and only if $f(c) = 0$.

Proof. If $x - c$ is a factor of $f(x)$, then $f(x) = (x - c)q(x)$ for some $q(x)$. Substituting $x = c$ gives

$$f(c) = (c - c)q(c) = 0 \cdot q(c) = 0.$$

If $f(c) = 0$, then the remainder when we divide $f(x)$ by $x - c$ is 0, so $x - c$ is a factor of $f(x)$. \square

Euler discovered that $n^2 + n + 41$ generates 40 consecutive primes for $0 \leq n \leq 39$. However, $40^2 + 40 + 41 = 41^2$. Euler's polynomial is prime for 86 of the first 100 natural numbers.

Example 2.44 (Legendre). Prove that there is no nonconstant polynomial with integral coefficients that produces primes for every integer n .

Proof. Suppose that there is such a polynomial with integer coefficients and $a_k \neq 0$:

$$f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_2 n^2 + a_1 n + a_0.$$

For a fixed value of n_0 , let $f(n_0) = p$ be a prime. For a variable integer t consider

$$\begin{aligned} f(n_0 + tp) &= a_k (n_0 + tp)^k + a_{k-1} (n_0 + tp)^{k-1} + \cdots + a_2 (n_0 + tp)^2 + a_1 (n_0 + tp) + a_0 \\ &\equiv a_k n_0^k + a_{k-1} n_0^{k-1} + \cdots + a_2 n_0^2 + a_1 n_0 + a_0 \\ &\equiv f(n_0) \equiv 0 \pmod{p}. \end{aligned}$$

Since f always generates primes, $f(n_0 + tp) = p$. However, then the polynomial $g(n) = f(n) - p$ has infinite roots, contradiction. Therefore there is no nonconstant prime producing polynomial. \square

While there is no prime-producing polynomial, in 1947, it was proven that there exists a positive real number r such that $\lfloor r^{3^n} \rfloor$ is prime for all natural numbers n . The value r is known as Mills' constant and its value is unknown, however, assuming the Riemann hypothesis, $r \approx 1.306378 \dots$

Example 2.45 (AIME). Find integers a and b such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$.

Solution. We know φ and ψ are the roots of $x^2 - x - 1$. Since $\varphi^2 = \varphi + 1$, we see that

$$\begin{aligned}\varphi^3 &= \varphi^2 + \varphi = 2\varphi + 1 \\ \varphi^4 &= \varphi^3 + \varphi^2 = 3\varphi + 2 \\ \varphi^5 &= \varphi^4 + \varphi^3 = 5\varphi + 3.\end{aligned}$$

These coefficients are Fibonacci numbers! We can prove by induction that $\varphi^n = F_n\varphi + F_{n-1}$. Since $x^2 - x - 1$ divides $ax^{17} + bx^{16} + 1$, φ must also be a root of $ax^{17} + bx^{16} + 1$. Therefore,

$$a\varphi^{17} + b\varphi^{16} + 1 = a(F_{17}\varphi + F_{16}) + b(F_{16}\varphi + F_{15}) + 1 = 0.$$

Hence, $aF_{17} + bF_{16} = 0$ and $aF_{16} + bF_{15} + 1 = 0$. We solve $a = F_{16} = \mathbf{987}$ and $b = -F_{17} = \mathbf{-1597}$. We can verify the second equation by Cassini's identity: $F_{16}^2 - F_{15}F_{17} = -1$. \square

Example 2.46 (Mandelbrot). Determine the positive integer a such that $x^8 + 5x^6 + 13x^4 + 20x^2 + 36$ is evenly divisible by $x^2 - x + a$.

Solution. Let $n(x) = x^8 + 5x^6 + 13x^4 + 20x^2 + 36$ and $d(x) = x^2 - x + a$, so

$$n(x) = d(x)q(x) \text{ for } q(x) \in \mathbb{Z}[x].$$

Since a is the constant term of $d(x)$, it must divide 36. Substituting $x = 1$ into the equation,

$$n(1) = d(1) \cdot q(1) \implies 75 = a \cdot q(1).$$

Since $a \mid 75$, $a \mid 36$, and a is positive, $a = 1$ or $a = 3$. Substituting $x = -2$ gives

$$n(-2) = d(-2) \cdot q(-2) \implies 900 = (a + 6) \cdot q(-2).$$

Since $7 \nmid 900$, we must have $a = \mathbf{3}$. Using long division, we verify

$$x^8 + 5x^6 + 13x^4 + 20x^2 + 36 = (x^2 - x + 3)(x^6 + x^5 + 3x^4 + 4x^2 + 4x + 12). \quad \square$$

Exercises

2.7.1. Find a constant c such that $x^4 + 3x^3 + cx^2 + 6x - 4$ is evenly divisible by $x - 2$.

2.7.2. Find the quotient $q(x)$ and remainder $r(x)$ upon dividing $x^6 + 2x^5 + 3x^3 + 1$ by $x^3 + x^2 - 1$.

2.7.3. Prove that if $d(x)$ is a factor of $f(x)$, then every root of $d(x)$ is also a root of $f(x)$.

§2.8 Review Problems

2.47. Prove that if n is an integer, then $1 + (-1)^n(2n - 1)$ is a multiple of 4.

2.48. (AMC 10) For each positive integer $m > 1$, let $P(m)$ denote the greatest prime factor of m . Find all n such that both $P(n) = \sqrt{n}$ and $P(n + 48) = \sqrt{n + 48}$.

2.49. Express $2^{22} + 1$ as the product of two four-digit numbers.

2.50. Find all positive integers $n < 17$ for which $n! + (n + 1)! + (n + 2)!$ is a multiple of 49.

2.51. (Duke) Find the sum of all integers n such that $n^2 + 2n + 2$ divides $n^3 + 4n^2 + 4n - 14$.

2.52. (ARML) Compute the largest prime factor of

$$3(3(3(3(3(3(3(3(3 + 1) + 1) + 1) + 1) + 1) + 1) + 1) + 1) + 1.$$

2.53. (Putnam) Determine the prime members of the sequence 101, 10101, 1010101, \dots .

2.54. Prove that the sum $1^k + 2^k + 3^k + \dots + n^k$ is divisible by $1 + 2 + 3 + \dots + n$ for all odd k .

2.55. Determine all values of n such that $1^2 + 2^2 + 3^2 + \dots + n^2$ is divisible by $1 + 2 + 3 + \dots + n$.

2.56. Find the remainder when $1^3 + 2^3 + 3^3 + \dots + 100^3$ is divided by 9.

2.57. (AHSME) If the base 8 representation of a perfect square is $ab3c$ where $a \neq 0$, find c .

2.58. (AHSME) Given $0 \leq x_0 < 1$, let

$$x_n = \begin{cases} 2x_{n-1} & \text{if } 2x_{n-1} < 1 \\ 2x_{n-1} - 1 & \text{if } 2x_{n-1} \geq 1 \end{cases}$$

for all integers $n > 0$. For how many x_0 is it true that $x_0 = x_5$?

2.59. (HMMT) Let Q be a polynomial $Q(x) = a_0 + a_1x + \dots + a_nx^n$, where a_0, \dots, a_n are non-negative integers. Given that $Q(1) = 4$ and $Q(5) = 152$, find $Q(6)$.

2.60. (AHSME) Let the product $(12)(15)(16)$, each factor written in base b , equal 3146 in base b . Let $s = 12 + 15 + 16$, each term expressed in base b . Find the value of s in base b .

2.61. Prove that every six-digit number of the form $abcabc$ is divisible by 7, 11, and 13.

2.62. (HMMT) What is the smallest 5-digit palindrome that is a multiple of 99?

2.63. Prove that every number in the sequence 16, 1156, 111556, \dots is a perfect square.

2.64. Prove that every positive integer m divides infinitely many Fibonacci numbers.

2.65. (AHSME) A set of consecutive positive integers beginning with 1 is written on a blackboard. When one number is erased, the arithmetic mean of the remaining numbers is $35\frac{7}{17}$. Find it.

2.66. (AMC 12) Call a number “prime-looking” if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?

2.67. Find the positive integer m such that the polynomial $p^3 + 2p + m$ divides $p^{12} - p^{11} + 3p^{10} + 11p^3 - p^2 + 23p + 30$.

§2.9 Challenge Problems

2.68. Fermat numbers are of the form $f_n = 2^{2^n} + 1$ for natural n .

- (a) $f_0 = 3, f_1 = 5, f_2 = 17, f_3 = 257$, and $f_4 = 65537$ are primes. Prove 641 divides $2^{32} + 1$.
(b) Prove that if $2^n + 1$ is prime for a positive integer n , then it is a Fermat number.

2.69* For an integer k in base 10, let $z(k)$ be the number of zeroes that appear in the binary representation of k . Let $S_n = \sum_{k=1}^n z(k)$. Compute S_{256} .

2.70* (AIME) Let $T = \{9^k : k \text{ is an integer, } 0 \leq k \leq 4000\}$. Given that 9^{4000} has 3817 digits and that its first (leftmost) digit is 9, how many elements of T have 9 as their leftmost digit?

2.71. (IMO) Let $f(n)$ denote the sum of the digits of n and $N = 4444^{4444}$. Find $f(f(f(N)))$.

2.72* (APMO) For any positive integer n , let $S(n)$ be the sum of digits in the decimal representation of n . Any positive integer obtained by removing one or more digits from the right-hand end of the decimal representation of n is called a stump of n . Let $T(n)$ be the sum of all stumps of n . Prove that $n = S(n) + 9T(n)$.

2.73* (USAMO) Prove that there is no polynomial P with integer coefficients such that there exists integers a, b, c with $P(a) = b, P(b) = c$, and $P(c) = a$.

2.74* Prove that for all positive integer pairs (a, b) with $b > 2$, $2^b - 1$ does not divide $2^a + 1$.

2.75* (IMO) Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1)$$

divides $abc - 1$.

2.76* (IMO) Find all pairs of positive integers such that $xy^2 + y + 7$ divides $x^2y + x + y$.

2.77* (Putnam) Let $0 < a_1 < a_2 < \dots < a_{mn+1}$ be $mn+1$ integers. Prove that you can find either $m+1$ of them no one of which divides any other, or $n+1$ of them, each dividing the following.

§A.2 Divisibility and Congruences Solutions

Exercises for Section 2.1

2.1.1 Observe that if 9 is amongst the digits, then the digits must sum to 18, however, this is impossible by parity. If 3 is included, then one of the other digits must be $1 \pmod 3$ and the other $2 \pmod 3$. Therefore, possible candidates are **315**, **135**, **375**, or **735**, however, $7 \nmid 375$. If 3 is excluded, then the number must be **175** or **715**, however $7 \nmid 715$. The answer is $\boxed{135, 175, 315, 735}$.

2.1.2 We complete the proof of Theorem 2.1:

(3) Since $dc \mid nc$, there exists an integer q with $nc = q(dc)$. Therefore $n = dq$, implying $d \mid n$.

(4) Since $d \mid n$ and $n \mid d$, there exists integers q and q' with $n = dq$ and $d = nq'$. Substituting,

$$n = nq'q \implies q'q = 1.$$

Therefore $|q'| = |q| = 1$ and we conclude $|n| = |d|$.

2.1.3 Factoring the polynomial using difference of squares,

$$\begin{aligned} n^5 - 5n^3 + 4n &= n(n^4 - 5n^2 + 4) \\ &= n(n^2 - 4)(n^2 - 1) \\ &= n(n - 2)(n + 2)(n - 1)(n + 1). \end{aligned}$$

Therefore, since it is the product of 5 consecutive integers, $n^5 - 5n^3 + 4n$ is divisible by $5! = 120$.

2.1.4 Since $d \mid n_j$ for every $1 \leq j \leq k$, there exists an integer q_j such that $n_j = q_j d$:

$$\begin{aligned} n_1\alpha_1 + n_2\alpha_2 + \cdots + n_k\alpha_k &= q_1d\alpha_1 + q_2d\alpha_2 + \cdots + q_kd\alpha_k \\ &= d \left(\sum_{1 \leq j \leq k} q_j\alpha_j \right). \end{aligned}$$

By the definition of divisibility, $d \mid n_1\alpha_1 + n_2\alpha_2 + \cdots + n_k\alpha_k$.

Exercises for Section 2.2

2.2.1 By the Sieve of Eratosthenes, Jane will erase every integer with an odd factor. Therefore, the numbers that remain are the powers of 2, 2^n . Since $2^{11} = 2048$, we must have $0 \leq n \leq 10$, for a total of $\boxed{11}$ values.

2.2.2 We factor $p^{1994} + p^{1995} = p^{1994}(1 + p)$. Since 1994 is even, $p + 1$ must be a perfect square:

$$p = n^2 - 1 = (n - 1)(n + 1), \text{ for integer } n.$$

For this to be a prime, we must have $n - 1 = 1 \implies n = 2$ and $p = \boxed{3}$.

2.2.3 We factor $N = 1001001001 = (10^3 + 1)(10^6 + 1)$. From the sum of cubes formula,

$$\begin{aligned} N &= [(10 + 1)(10^2 - 10 + 1)] [(10^2 + 1)(10^4 - 10^2 + 1)] \\ &= (11 \cdot 91)(101 \cdot 9901) \\ &= 7 \cdot 11 \cdot 13 \cdot 101 \cdot 9901. \end{aligned}$$

The largest divisor less than 10000 is $\boxed{9901}$.

2.2.4 Let $n = dq$ for some integer q . Then,

$$\begin{aligned} x^n - y^n &= x^{dq} - y^{dq} \\ &= (x^d)^q - (y^d)^q \\ &= (x^d - y^d)(x^{d(q-1)} + x^{d(q-2)}y^d + \dots + y^{d(q-1)}). \end{aligned}$$

Therefore, $x^d - y^d \mid x^n - y^n$ when $d \mid n$.

Exercises for Section 2.3

2.3.1 Let the three digit integer be $N = 13K + 7$. We solve the inequality:

$$13K + 7 < 1000 \implies 13K < 993 \implies K < 77.$$

Therefore, the largest N is when $K = 76$, giving $N = 13 \cdot 76 + 7 = \boxed{995}$.

2.3.2 We group the numbers in the list based upon their remainder when dividing by 7:

$$\begin{aligned} 7k &: \{7, 14, 21, 28, 35, 42, 49\} \\ 7k + 1 &: \{1, 8, 15, 22, 29, 36, 43, 50\} \\ 7k + 2 &: \{2, 9, 16, 23, 30, 37, 44\} \\ 7k + 3 &: \{3, 10, 17, 24, 31, 38, 45\} \\ 7k + 4 &: \{4, 11, 18, 25, 32, 39, 46\} \\ 7k + 5 &: \{5, 12, 19, 26, 33, 40, 47\} \\ 7k + 6 &: \{6, 13, 20, 27, 34, 41, 48\}. \end{aligned}$$

We can have a maximum of **1** number that is $7k$, **8** numbers that are $7k + 1$, **7** numbers that are $7k + 2$, and **7** numbers that are $7k + 3$. The answer is hence $1 + 8 + 7 + 7 = \boxed{23}$.

2.3.3 From the division algorithm, $2017 = dq + r$ for $0 \leq r \leq d - 1$. We condition on the quotient.

- If $q = 1$, then $2017 = d + r \leq 2d - 1 \implies d \geq 1009$. However, d is at most 1000.
- If $q = 2$, then $2017 = 2d + r \leq 3d - 1 \implies d \geq 673$. For $d = 673$, we see that

$$2017 = 673 \cdot 2 + \mathbf{671}.$$

As the quotient increases, the remainder decreases. Therefore, the largest remainder is $r = \boxed{671}$.

2.3.4 Let the two numbers be $4m + 3$ and $4n + 3$. Expanding their product, we see

$$(4m + 3)(4n + 3) = 16mn + 12m + 12n + 9 = 4(4mn + 3m + 3n + 2) + 1,$$

which is of the form $4k + 1$.

Exercises for Section 2.4

2.4.1 Since $a_1 \equiv a_2 \pmod{m}$, by definition, $m \mid a_1 - a_2$. Furthermore, $d \mid m$, therefore by the transitive property of divisibility, $d \mid a_1 - a_2$. This then implies $a_1 \equiv a_2 \pmod{d}$.

2.4.2 (a) Note that $5^3 = 125 \equiv 1 \pmod{31}$. Therefore, since $1337 \equiv 2 \pmod{3}$,

$$5^{1337} \equiv 5^2 \equiv \boxed{25} \pmod{31}.$$

(b) Note that $2^7 \equiv 1 \pmod{2^7 - 1}$. Therefore, since $2001 \equiv 6 \pmod{7}$,

$$2^{2001} \equiv 2^6 \equiv \boxed{64} \pmod{2^7 - 1}.$$

2.4.3 Rewriting the expression and using the Binomial Theorem,

$$\begin{aligned} 2^{4n} + 10n - 1 &= 16^n + 10n - 1 \\ &= (15 + 1)^n + 10n - 1 \\ &= \left[15^n + \binom{n}{1} 15^{n-1} + \cdots + \binom{n}{n-2} 15^2 + \binom{n}{n-1} 15 + 1 \right] + 10n - 1 \\ &\equiv [0 + 15n + 1] + 10n - 1 \equiv 0 \pmod{25}. \end{aligned}$$

2.4.4 We use our difference of n th powers factorization:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}).$$

Since $a \equiv b \pmod{n}$, the first term in the product is divisible by n . Furthermore,

$$a^{n-1} + a^{n-2}b + \cdots + b^{n-1} \equiv \underbrace{b^{n-1} + b^{n-1} + \cdots + b^{n-1}}_{n \text{ terms}} \equiv 0 \pmod{n}.$$

Therefore, the second term is also divisible by n , implying the product is divisible by n^2 .

Exercises for Section 2.5

2.5.1 In base 3, $2000 = 2 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^3 + 2$. Observe that $2 \cdot 3^k = 3^{k+1} - 3^k$, so

$$2000 = (3^7 - 3^6) + (3^6 - 3^5) + (3^4 - 3^3) + (3 - 1) = 3^7 - 3^5 + 3^4 - 3^3 + 3 - 1.$$

Therefore, the sum of exponents used is $7 + 5 + 4 + 3 + 1 + 0 = \boxed{20}$.

2.5.2 Let the base be b . Using difference of squares,

$$\begin{aligned} 10101_b &= b^4 + b^2 + 1 \\ &= (b^2 + 1)^2 - b^2 \\ &= (b^2 - b + 1)(b^2 + b + 1). \end{aligned}$$

Since $b \geq 2$, both factors are at least 3, therefore, the number is composite.

2.5.3 WLOG say $x > y > z$. For $x = k$, we see that $y, z \in \{0, 1, \dots, k-1\}$ for a total of $\binom{k}{2}$ ways to choose y and z . We repeat this procedure until $x = 8$, giving the sum

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \dots + \binom{8}{2} = 84.$$

Therefore, $(x, y, z) = (8, 7, 6)$ is the 84th element of the set. Now, we desire to find the 16th smallest element for $x = 9$. We see that for $y = 1$ to $y = 5$, there are $1 + 2 + 3 + 4 + 5 = 15$ pairs. Therefore, the 100th smallest element is $(x, y, z) = (9, 6, 0)$ giving $2^9 + 2^6 + 2^0 = \boxed{577}$.

2.5.4 Using the formula for a sum of an infinite geometric series,

$$\begin{aligned} \frac{n}{810} &= \frac{100d + 25}{10^3} + \frac{100d + 25}{10^6} + \dots \\ &= \frac{100d + 25}{999}. \end{aligned}$$

We isolate the equation for n and simplify:

$$n = \frac{810}{999} (100d + 25) = \frac{30}{37} \cdot 25 (4d + 1).$$

Hence $37 \mid 4d + 1$, so $d = 9$. Then, $n = 30 \cdot 25 = \boxed{750}$.

Exercises for Section 2.6

2.6.1 Let such a five-digit number be $abcde$. Since $43 = 5 \cdot 9 - 2$, the digits must either consist of 4 nines and 1 seven or 3 nines and 2 eights. Therefore, there are $\binom{5}{1} + \binom{5}{2} = 15$ such numbers. By the divisibility rule for 11, $a - b + c - d + e$ must be divisible by 11. The maximum value is

$$a - b + c - d + e = 2(a + c + e) - (a + b + c + d + e) \leq 2 \cdot 27 - 43 = 11,$$

with equality holding when $a = b = c = 9$. Therefore, we must have $(b, d) = (7, 9), (8, 8), (9, 7)$ for a total of 3 numbers that are divisible by 11. The probability is $3/15 = \boxed{1/5}$.

2.6.2 Let the decimal form be $N = a_m a_{m-1} \dots a_1 a_0$. Reducing mod 2^k , we see

$$\begin{aligned} N &= 10^m a_m + 10^{m-1} a_{m-1} + \dots + 10^1 a_1 + a_0 \\ &\equiv 10^{k-1} a_{k-1} + 10^{k-2} a_{k-2} + \dots + a_1 + a_0 \\ &\equiv a_{k-1} a_{k-2} a_{k-3} \dots a_1 a_0 \pmod{2^k}. \end{aligned}$$

Therefore, n is divisible by 2^k if and only if the last k digits are.

2.6.3 (a) Let $N = 10a + x$ have units digit x . Multiplying by 4,

$$4N = 40a + 4x \equiv a + 4x \pmod{13}.$$

Therefore, we conclude that 13 divides N if and only if 13 divides $a + 4x$.

(b) We desire to find a prime p such that $p \mid 10a + x$ implies $p \mid a - 3x$. Note

$$10a + x - 10(a - 3x) = 31x,$$

therefore $p \mid 31$ by the linear combination theorem. Hence, $p = \boxed{31}$.

2.6.4 Since $1001 = 7 \cdot 11 \cdot 13$, we see $10^3 \equiv -1 \pmod{91}$. Now,

$$\begin{aligned} 123450789 &= 123 \cdot 10^6 + 450 \cdot 10^3 + 789 \\ &\equiv 123 \cdot 1 + 450 \cdot -1 + 789 \cdot 1 \\ &\equiv 32 + 5 + 61 \\ &\equiv 98 \equiv 7 \pmod{91}. \end{aligned}$$

Therefore, $12345N789 \equiv 7 + 1000N \equiv 7 - N \equiv 0 \pmod{91} \implies N = \boxed{7}$.

Exercises for Section 2.7

2.7.1 Let $f(x) = x^4 + 3x^3 + cx^2 + 6x - 4$. By the factor theorem, we must have $f(2) = 0$, hence

$$f(2) = 16 + 24 + 4c + 12 - 4 = 0 \implies 4c + 48 = 0 \implies c = \boxed{-12}.$$

2.7.2 We divide the leading terms of the two polynomials: $x^6/x^3 = x^3$. Hence,

$$x^6 + 2x^5 + 3x^3 + 1 = (x^3 + x^2 - 1) \cdot x^3 + (x^5 + 4x^3 + 1)$$

To eliminate the quintic term, we divide x^5 by x^3 : $x^5/x^3 = x^2$ and add to our quotient:

$$x^6 + 2x^5 + 3x^3 + 1 = (x^3 + x^2 - 1) \cdot (x^3 + x^2) + (-x^4 + 4x^3 + x^2 + 1)$$

To remove the quartic term, we divide $-x^4$ by x^3 : $-x^4/x^3 = -x$ and add to our quotient:

$$x^6 + 2x^5 + 3x^3 + 1 = (x^3 + x^2 - 1) \cdot (x^3 + x^2 - x) + (5x^3 + x^2 - x + 1)$$

To remove the cubic term, we divide $5x^3$ by x^3 : $5x^3/x^3 = 5$ and add to our quotient:

$$x^6 + 2x^5 + 3x^3 + 1 = (x^3 + x^2 - 1) \cdot (x^3 + x^2 - x + 5) + (-4x^2 - x + 6)$$

Therefore, $q(x) = \boxed{x^3 + x^2 - x + 5}$ and $r(x) = \boxed{-4x^2 - x + 6}$.

2.7.3 Since $d(x) \mid f(x)$, $f(x) = d(x)q(x)$. Let r be an arbitrary root of $d(x)$. Then,

$$f(r) = d(r) \cdot q(r) = 0 \cdot q(r) = 0,$$

thus r is also a root of $f(x)$.

Review Problems

2.47 The value of $(-1)^n$ depends on the parity of n . Therefore, we have two cases:

(i) n is even. Then, $n = 2k$ for some integer k . Substituting this into the expression:

$$\begin{aligned} 1 + (-1)^n (2n - 1) &= 1 + (-1)^{2k} (2 \cdot 2k - 1) \\ &= 1 + 1(4k - 1) \\ &= 4k. \end{aligned}$$

(ii) n is odd. Then, $n = 2k + 1$ for some integer k . Substituting this into the expression:

$$\begin{aligned} 1 + (-1)^n (2n - 1) &= 1 + (-1)^{2k+1} (2 \cdot (2k + 1) - 1) \\ &= 1 - (4k + 2 - 1) \\ &= -4k. \end{aligned}$$

2.48 We see that $n = p^2$ and $n + 48 = q^2$ for primes p and q . By difference of squares,

$$q^2 - p^2 = (q + p)(q - p) = 48.$$

If $q + p = 24$ and $q - p = 2$, then $(q, p) = (13, 11)$. For $q + p = 12$ and $q - p = 4$, $(q, p) = (8, 4)$ and for $q + p = 8$ and $q - p = 6$, $(q, p) = (7, 1)$. Neither are prime solutions, hence $n = 11^2 = \boxed{121}$.

2.49 We complete the square:

$$\begin{aligned} 2^{22} + 1 &= (2^{11} + 1)^2 - (2^6)^2 \\ &= (2^{11} + 1 - 2^6)(2^{11} + 1 + 2^6) \\ &= 1985 \cdot 2113. \end{aligned}$$

The desired four-digit numbers are $\boxed{1985}$ and $\boxed{2113}$.

2.50 Factoring, we see

$$n! + (n + 1)! + (n + 2)! = n! [1 + (n + 1) + (n + 1)(n + 2)] = n!(n + 2)^2.$$

The expression is divisible by 49 if and only if $7 \mid n + 2$ or $49 \mid n!$. The first condition happens when $n = 5$ or $n = 12$ and the second when $n \geq 14$. Therefore, the answers are $\boxed{5, 12, 14, 15, 16}$.

2.51 From the division algorithm for polynomials,

$$n^3 + 4n^2 + 4n - 14 = (n^2 + 2n + 2)(n + 2) + (-2n - 18).$$

When $n^2 + 2n + 2$ divides $n^3 + 4n^2 + 4n - 14$, it must also divide the remainder above, $-2n - 18$. If $|-2n - 18| \geq |n^2 + 2n + 2|$, then $-4 \leq n \leq 4$. Testing values shows $n = -4, -2, -1, 0, 1, 4$. Alternatively, if $-2n - 18 = 0$, then $n = -9$. The sum of all n is hence $\boxed{-11}$.

2.52 Using the sum of a geometric series formula and factoring,

$$\begin{aligned} 3^{11} + 3^{10} + 3^9 + \cdots + 3^0 &= \frac{3^{12} - 1}{2} \\ &= \frac{(3^6 - 1)(3^6 + 1)}{2} \\ &= \frac{(3^3 - 1)(3^3 + 1)(3^6 + 1)}{2} \\ &= \frac{26 \cdot 28 \cdot 730}{2}. \end{aligned}$$

Since $26 = 2 \cdot 13$, $28 = 2^2 \cdot 7$, and $730 = 2 \cdot 5 \cdot 73$, the largest prime factor is $\boxed{73}$.

2.53 Let k_n represent the member of the sequence with n 1's. From grouping terms, we see that 101 divides k_{2n} , therefore, the only prime with an even index is $\boxed{101}$. Furthermore,

$$\begin{aligned} k_{2n+1} &= 1 + 10^2 + 10^4 + \cdots + 10^{4n} = \frac{10^{4n+2} - 1}{99} \\ &= \left(\frac{10^{2n+1} + 1}{11} \right) \left(\frac{10^{2n+1} - 1}{9} \right). \end{aligned}$$

Since both of these are integers greater than 1, no member of the sequence is prime for odd indices.

2.54 Recall that $1 + 2 + \cdots + n = n(n+1)/2$. Let S be the sum of the odd k^{th} powers. Multiplying the divisibility by 2 and using the converse of Example 2.2, we must prove $n \mid 2S$ and $n+1 \mid 2S$. To prove $n+1$ divides the sum, we group it into pairs that sum to $n+1$:

$$2S = \left(1^k + n^k\right) + \cdots + \left(d^k + (n+1-d)^k\right) + \cdots + \left(n^k + 1\right).$$

To prove n divides the sum, we group it into pairs that add to n :

$$2S = \left(1^k + (n-1)^k\right) + \cdots + \left(d^k + (n-d)^k\right) + \cdots + \left((n-1)^k + 1\right) + 2n^k.$$

Since $x+y \mid x^k + y^k$ for odd k , we have proven $n \mid 2S$ and $n+1 \mid 2S$.

2.55 Recall the sum of squares formula $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$. Therefore,

$$n(n+1)/2 \mid n(n+1)(2n+1)/6.$$

Using the cancellation property, this is equivalent to $3 \mid 2n+1$, hence $n \equiv \boxed{1 \pmod{3}}$.

2.56 Clearly $(3n)^3 \equiv 0 \pmod{9}$. Using the Binomial Theorem,

$$\begin{aligned} (3n+1)^3 &= 27n^3 + 27n^2 + 9n + 1 \equiv 1 \pmod{9} \\ (3n+2)^3 &= 27n^3 + 54n + 36n + 8 \equiv 8 \pmod{9}. \end{aligned}$$

Therefore, every group of 3 sums to 0 mod 9. Hence,

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + 99^3 + 100^3 &= \sum_{k=0}^{32} [(3k+1)^3 + (3k+2)^3 + (3k+3)^3] + 100^3 \\ &\equiv 0 + 1 \equiv 1 \pmod{9}. \end{aligned}$$

Another method uses the sum of cubes formula:

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + 100^3 &= \left(\frac{100 \cdot 101}{2} \right)^2 \\ &= 50^2 \cdot 101^2 \\ &\equiv 5^2 \cdot 2^2 \equiv 10^2 \equiv 1 \pmod{9}. \end{aligned}$$

2.57 Converting to decimal,

$$ab3c_8 = 512a + 64b + 24 + c = 8(64a + 8b + 3 + c).$$

If c is even, then $64a + 8b + 3 + c$ is odd. However, 8 only has three factors of 2, hence $ab3c_8$ cannot be a perfect square. Furthermore, we see all odd perfect squares are 1 mod 8. Indeed,

$$(2n+1)^2 = 4n^2 + 4n + 1 = 4n(n+1) + 1.$$

Since one of n and $n+1$ is even, this entire quantity is 1 mod 8. Therefore, $c = \boxed{1}$.

2.58 In binary, the given recursion is equivalent to right shifting the decimal by 1. Let

$$x_0 = \sum \frac{a_k}{2^k} = .(a_1a_2a_3a_4a_5 \cdots)_2.$$

If $a_1 = 0$, then $x_1 = 2x_0$ and if $a_1 = 1$, then $x_1 = 2x_0 - 1$. Either way, $x_1 = .(a_2a_3a_4a_5a_6 \cdots)_2$. To determine x_5 , we right shift by 5 bits: $x_5 = .(a_6a_7a_8a_9a_{10} \cdots)_2$. Since $x_0 = x_5$, x_0 must repeat with $a_1a_2a_3a_4a_5$ as the repeating block. There are 2 choices per digit, so we have $2^5 = 32$ blocks. However, if $a_1 = a_2 = a_3 = a_4 = a_5 = 1$, then $x_0 = 1$, so the answer is $32 - 1 = \boxed{31}$.

2.59 Since $Q(1) = 4$, the sum of the coefficients equals 4. Converting 152 to base 5, we see $152 = 1102_5$, so $Q(x) = x^3 + x^2 + 2$. Therefore, $Q(6) = 216 + 36 + 2 = \boxed{254}$.

2.60 We are given that $12_b \cdot 15_b \cdot 16_b = 3146_b$. Converting back to decimal,

$$\begin{aligned} (b+2)(b+5)(b+6) &= 3b^3 + b^2 + 4b + 6 \\ b^3 + 13b^2 + 52b + 60 &= 3b^3 + b^2 + 4b + 6 \\ 0 &= 2b^3 - 12b^2 - 48b - 54 \end{aligned}$$

Dividing by 2, $b^3 - 6b^2 - 24b - 27 = 0$. Since the digit 6 is used, $b \geq 7$. We see $b = 9$ is the only real root of the cubic, therefore $s = 12_9 + 15_9 + 16_9 = \boxed{44_9}$.

2.61 Let N be such a number. Grouping the terms,

$$\begin{aligned} n = abcabc &= 10^5 \cdot a + 10^4 \cdot b + 10^3 \cdot c + 10^2 \cdot a + 10^1 \cdot b + c \\ &= (10^5 + 10^2) \cdot a + (10^4 + 10^1) \cdot b + (10^3 + 1) \cdot c \\ &= 1001 \cdot (100a + 10b + c). \end{aligned}$$

Since $1001 = 7 \cdot 11 \cdot 13$, we conclude that $abcabc$ is divisible by 7, 11, and 13.

2.62 Observe that a 5-digit palindrome is of the form

$$XYZYX = 10000X + 1000Y + 100Z + 10Y + X = 10001X + 1010Y + 100Z.$$

Notice $99 \mid 9999$ and $99 \mid 990$, so $10001 \equiv 2 \pmod{99}$ and $1010 \equiv 20 \pmod{99}$. Therefore,

$$10001X + 1010Y + 100Z \equiv 2X + 20Y + Z \equiv 0 \pmod{99}.$$

Since X, Y , and Z are all digits, $0 \leq X, Y, Z \leq 9$, so $2X + 20Y + Z = 99$ or $2X + 20Y + Z = 198$.

- In the first case, the smallest palindrome we obtain is when $Y = 4$ giving $2X + Z = 19$. We must have $Z = 9 \implies X = 5$, giving the palindrome 54945.
- In the second case, the smallest palindrome we obtain is when $Y = 9$ giving $2x + Z = 18$. We must have $Z = 8 \implies X = 5$, giving the palindrome 59895.

The smaller of the two cases is $\boxed{54945}$.

2.63 By the geometric series formula, we see that a sequence k 1's equals

$$\underbrace{11 \cdots 111}_{k \text{ 1's}} = 10^{k-1} + 10^{k-2} + \cdots + 10^2 + 10^1 + 1 = \frac{10^k - 1}{9}.$$

Therefore, we can write every number in the sequence in the form

$$\begin{aligned} \underbrace{11 \cdots 111}_{k \text{ 1's}} \underbrace{55 \cdots 55}_{k-1 \text{ 1's}} 6 &= \frac{10^k (10^k - 1)}{9} + \frac{50 (10^{k-1} - 1)}{9} + 6 \\ &= \frac{10^{2k} + 4 \cdot 10^k + 4}{9} \\ &= \left(\frac{10^k + 2}{3} \right)^2. \end{aligned}$$

Since $10^k + 2 \equiv 1 + 2 \equiv 0 \pmod{3}$, every term in the sequence is the square of an integer.

2.64 We make a list of the possible pairs of consecutive Fibonacci numbers, (F_{k+2}, F_{k+1}) for nonnegative integers k . This list is infinite, but reduced modulo m , there are only m^2 distinct pairs. Therefore, by the pigeonhole principle, two pairs must be equal. Then, by induction, we see that the entire sequence is periodic modulo m . Hence, there exists an integer $d > 2$ such that

$$(F_{d+2}, F_{d+1}) \equiv (1, 1) \pmod{m} \implies F_d \equiv F_{d+2} - F_{d+1} \equiv 0 \pmod{m}.$$

We know $F_d \mid F_{qd}$, hence $F_{qd} \equiv 0 \pmod{m}$ for infinite values of q .

2.65 Let the number of consecutive integers be n and $1 \leq k \leq n$ be the erased number. The smallest possible arithmetic mean occurs when $k = n$ is removed, giving an average of $\frac{n}{2}$. The largest occurs when $k = 1$ is removed, giving an average of $\frac{n+2}{2}$. We have the inequality

$$\frac{n}{2} \leq 35\frac{7}{17} \leq \frac{n+2}{2} \implies n \leq 70\frac{14}{17} \leq n+2.$$

Since we average $n - 1$ numbers after removing one, $17 \mid n - 1$, hence $n = 69$. Therefore,

$$\frac{1/2 \cdot 69 \cdot 70 - k}{68} = 35\frac{7}{17} \implies k = \boxed{7}.$$

2.66 Let S_n denote the numbers ≤ 1000 divisible by n . We divide the numbers into five sets:

$$\{1\}, \{S_2\}, \{S_3\}, \{S_5\}, \{\text{primes not including } 2, 3, 5\}, \{\text{prime-looking}\}.$$

By complimentary counting, the number of prime-looking numbers is $1000 - 165 - 1 - |S_2 \cup S_3 \cup S_5|$. Notice that 2, 3, and 5 are prime. We calculate $S_2 \cup S_3 \cup S_5$ using PIE:

$$\begin{aligned} |S_2 \cup S_3 \cup S_5| &= \lfloor \frac{1000}{2} \rfloor + \lfloor \frac{1000}{3} \rfloor + \lfloor \frac{1000}{5} \rfloor - \lfloor \frac{1000}{6} \rfloor - \lfloor \frac{1000}{10} \rfloor - \lfloor \frac{1000}{15} \rfloor + \lfloor \frac{1000}{30} \rfloor \\ &= 500 + 333 + 200 - 166 - 100 - 66 + 33 = 734. \end{aligned}$$

The number of prime-looking numbers is hence $1000 - 165 - 1 - 734 = \boxed{100}$.

Challenge Problems

2.68 (a) Observe that $2^{16} = 65536 \equiv 154 \pmod{641}$. Therefore,

$$2^{32} + 1 = 65536^2 + 1 \equiv 154^2 + 1 = 23717 = 641 \cdot 37 \equiv 0 \pmod{641}.$$

An alternative proof involves manipulating the equation $641 = 2^7 \cdot 5 + 1 = 2^4 + 5^4$.

(b) Suppose $n = 2^r \cdot s$ for some nonnegative integer r and positive odd integer $s > 1$. Then,

$$2^n + 1 = (2^{2^r})^s + 1 = (2^{2^r} + 1)(2^{2^r \cdot (s-1)} - \dots + 2^{2^r \cdot 2} - 2^{2^r} + 1).$$

However, then $2^n + 1$ is composite. Thus, n cannot have an odd prime factor, so $n = 2^r$.

2.69 The three-digit binary numbers are $100_2, 101_2, 110_2$, and 111_2 . After the first digit, the rest of the digits are evenly split between zeros and ones, hence there are $2 \cdot \frac{1}{2} \cdot 2^2 = 4$ zeros. In general, for an m -digit binary number, there are $m - 1$ digits after the initial 1 for a total of 2^{m-1} integers. The proportion of zeroes for every digit is $\frac{1}{2}$, hence we have $(m - 1) \cdot \frac{1}{2} \cdot 2^{m-1}$ zeros in total. Therefore,

$$S_{255} = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 4 + 4 \cdot 8 + 5 \cdot 16 + 6 \cdot 32 + 7 \cdot 64 = 769.$$

Adding the 8 zeroes in 256, our answer is $S_{256} = 769 + 8 = \boxed{777}$.

2.70 Note 9^{4000} has 3816 more digits than 9^0 . Furthermore, 9^n has one more digit than 9^{n-1} except in the case that 9^n starts with a 9 (implying 9^{n-1} starts with a 1). Hence, computing 9^n from 9^{n-1} for $1 \leq n \leq 4000$, there are $4000 - 3816 = \boxed{184}$ instances when the digits do not increase. Since $9^0 = 1$, this is exactly the number of elements of T with leading digit 9.

2.71 For every positive integer n , $f(n) \equiv n \pmod{9}$. Therefore,

$$f(f(f(n))) \equiv f(f(n)) \equiv f(n) \equiv n \pmod{9}.$$

We desire to compute $4444^{4444} \pmod{9}$. Since $4444 \equiv 7 \pmod{9}$ and $7^3 \equiv 1 \pmod{9}$,

$$4444^{4444} \equiv 7^{4444} \equiv 7^1 \equiv 7 \pmod{9}.$$

Notice $4444^{4444} < 10^{4 \cdot 4444} = 10^{17776}$. Therefore, 4444^{4444} has at most 17776 digits, so

$$f(n) < 9 \cdot 17776 = 159984.$$

Furthermore, $f(f(n))$ is maximized when $f(n) = 99999$ giving $f(f(n)) = 9 \cdot 5 = 45$. Finally, $f(f(f(n))) \leq f(39) = 12$. Since $f(f(f(n))) \equiv 7 \pmod{9}$, we must have $f(f(f(N))) = \boxed{7}$.

2.72 Let $n = d_k d_{k-1} \cdots d_2 d_1 d_0 = \sum_{i=0}^k 10^i d_i$, where d_i are the digits of n . Observe that the result of removing the first m digits is $U_m(n) = \sum_{i=m}^k 10^{i-m} d_i$. Therefore,

$$T(n) = \sum_{m=1}^k U_m(n) = \sum_{m=1}^k \sum_{i=m}^k 10^{i-m} d_i.$$

By the geometric series formula, the coefficient of an arbitrary digit d_j is

$$10^{j-1} + 10^{j-2} + \cdots + 10 + 1 = \frac{10^j - 1}{9}.$$

Therefore, $T(n) = \sum_{j=0}^k \left(\frac{10^j - 1}{9} \right) d_j$. Finally, we see that

$$S(n) + 9T(n) = \sum_{j=0}^k \left(1 + 9 \cdot \frac{10^j - 1}{9} \right) d_j = \sum_{j=0}^k 10^j d_j = n.$$

2.73 We begin by proving that for integers m and n , $m - n \mid P(m) - P(n)$.

Proof. Let $P(x) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_1 x + a_0$. Then,

$$\begin{aligned} P(m) - P(n) &= (a_r m^r + a_{r-1} m^{r-1} + \cdots + a_1 m + a_0) - (a_r n^r + a_{r-1} n^{r-1} + \cdots + a_1 n + a_0) \\ &= a_r (m^r - n^r) + a_{r-1} (m^{r-1} - n^{r-1}) + \cdots + a_1 (m - n). \end{aligned}$$

Since $m - n \mid m^s - n^s$ for all exponents s , the identity is established. \square

Assume for the sake of contradiction there exists such a polynomial. Then,

$$a - b \mid P(a) - P(b) = b - c \mid P(b) - P(c) = c - a \mid P(c) - P(a) = a - b.$$

Therefore $|a - b| = |b - c| = |c - a| = k$ for some constant k . However,

$$0 = (a - b) + (b - c) + (c - a) = mk,$$

where m is one of $-3, -1, 1$, or 3 . Hence $k = 0$, implying $a = b = c$, contradiction.

2.74 Assume that $2^b - 1 \mid 2^a + 1$ for some integer pair (a, b) . Since $a > b > 2$, we write $a = bq + r$, where $0 \leq r \leq b - 1$ using the division algorithm. Furthermore, $2^b \equiv 1 \pmod{2^b - 1}$, therefore

$$2^a \equiv 2^{bq} 2^r \equiv 2^r \equiv -1 \pmod{2^b - 1} \implies 2^b - 1 \mid 2^r + 1.$$

However, $2^r + 1 \leq 2^{b-1} + 1 < 2^b - 1$ for $b > 2$, therefore, this is a contradiction.

2.75 Set $x = a - 1, y = b - 1$, and $z = c - 1$ where $1 \leq x \leq y \leq z$. Therefore,

$$(x + 1)(y + 1)(z + 1) - 1 = xyz + yz + xz + xy + z + y + x.$$

Let $f(x, y, z)$ be the quotient of divisibility. Expanding and simplifying, we see that

$$f(x, y, z) = \frac{yz + xz + xy + z + y + x}{xyz} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xy} + \frac{1}{xz} + \frac{1}{yz}.$$

Furthermore, f is decreasing in the variables x, y, z , hence the maximum value is

$$f(1, 2, 3) = 2\frac{5}{6} \implies f(x, y, z) \in \{1, 2\}.$$

Furthermore, $f(3, 4, 5) = \frac{59}{60} < 1$, therefore x is either 1 or 2. We finish with casework.

If $x = 1$, then $f(1, y, z) = 1 + \frac{1}{y} + \frac{1}{z} + \frac{1}{y} + \frac{1}{z} + \frac{1}{yz} = 2$. Multiplying by yz and using SFFT,

$$yz = 2y + 2z + 1 \implies (y - 2)(z - 2) = 5 \implies (y, z) = (3, 7).$$

If $x = 2$, since $f(2, 3, 4) = \frac{35}{24} < 2$, we must have

$$f(2, y, z) = \frac{1}{2} + \frac{1}{y} + \frac{1}{z} + \frac{1}{2y} + \frac{1}{2z} + \frac{1}{yz} = 1.$$

Multiplying by $2yz$ and simplifying using Simon's Favourite Factoring Trick,

$$yz = 3y + 3z + 2 \implies (y - 3)(z - 3) = 11 \implies (y, z) = (4, 14).$$

Adding 1 to the triples found above gives $(a, b, c) = \boxed{(2, 4, 8), (3, 5, 15)}$.

2.76 We find a linear combination of $xy^2 + y + 7$ and $x^2y + x + y$ that cancels several terms:

$$y(x^2y + x + y) - x(xy^2 + y + 7) = y^2 - 7x.$$

Hence $xy^2 + y + 7 \mid y^2 - 7x$. Since the LHS is positive, we have three cases:

- $y^2 - 7x > 0$. We arrive at a contradiction since $xy^2 + y + 7 > y^2 > y^2 - 7x$.
- $y^2 - 7x = 0$. We have the solution $(x, y) = (7m^2, 7m)$ for some positive integer m .
- $y^2 - 7x < 0$. Since $7x - y^2$ is positive, we multiply by -1 :

$$xy^2 + y + 7 \mid y^2 - 7x \implies xy^2 + y + 7 \mid 7x - y^2.$$

Now, $xy^2 + y + 7 \leq 7x - y^2$. Comparing the coefficient of x shows $y^2 \leq 7 \implies y = 1, 2$.

- If $y = 1$, then $x + 8 \mid 7x - 1$. Since $7(x + 8) - (7x - 1) = 57$, we see $x + 8 \mid 57$. The divisors of 57 are 1, 3, 19, 57, so $x = 11$ or $x = 49$ yielding $(x, y) = (\mathbf{11}, \mathbf{1}), (\mathbf{49}, \mathbf{1})$.
- If $y = 2$, then $4x + 9 \mid 7x - 4$. However, $2(4x + 9) = 8x + 18 > 7x - 4$, therefore, the quotient must be 1. However, $4x + 9 = 7x - 4$ has no integer solutions.

The solutions are hence $(x, y) = \boxed{(11, 1), (49, 1), (7m^2, 7m)}$.

2.77 Assume neither sequence exists. We poset the sequence using the ordering

$$a_m \prec a_n \leftrightarrow a_m < a_n, a_m \mid a_n.$$

Define a function $f : \mathcal{N}_{mn+1} \rightarrow \mathcal{N}_m \times \mathcal{N}_n$ with $f(t) = (x_t, y_t)$, where x_t is the largest chain beginning with a_t , while y_t is the largest decreasing anti-chain beginning with a_t . There are mn possible values of f , hence by the pigeonhole principle, there exists indices $j < k$ with $f(j) = f(k)$.

However, if $a_j \mid a_k$, then $x_j \geq x_k + 1$. Otherwise, if $a_j \nmid a_k$, then $a_k \succ a_j$, hence $y_k \geq y_j + 1$. Both inequalities contradict $f(j) = f(k)$, therefore, our original assumption was false. Hence, we can find either a chain of length $m + 1$ or an anti-chain of length $n + 1$.

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