Riemann Zeta Function

$\zeta(s)$ for integer values of s

Justin Stevens

Outline

Riemann Zeta Definition

Definition. The Riemann zeta function, $\zeta(s)$, is a function of a complex variable s that analytically continues the sum of the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots, \quad \operatorname{Re}(s) > 1.$$

The Riemann zeta function plays a pivotal role in analytical number theory and has applications in physics, probability theory, and applied statistics.

By the Fundamental Theorem of Arithmetic and distributive property,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right)$$

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$$\zeta(s) = \prod_{\text{prime } p} \left(\frac{1}{1 - 1/p^s}\right).$$

On the Number of Primes Less Than a Given Magnitude

VII.

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1 - \frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\,\zeta(s) = \int_0^\infty \frac{x^{s-1}\,dx}{e^x - 1}.$$

Basel Problem

Euler computed the value of $\zeta(2)$ in 1734 at the age of twenty-eight. In doing so, he solved the Basel problem, named after the hometown of Euler

Dividing the Maclaurian series expansion of the sine function by x,

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots .$$

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$$-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{6} \implies \zeta(2) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Probability of Random Numbers Being Coprime

Example. Compute the probability two random integers are coprime.

Solution. The probability a random number is divisible by a prime p is $1/p$, thus the probability two numbers are both divisible by p is $1/p^2$. Hence

$$P(\text{Two Numbers Not Both Divisible by } p) = 1 - \frac{1}{p^2}.$$

Two numbers are coprime if and only if they share no prime divisors, thus

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Wallis Product

Recall $\frac{\sin(x)}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right)$. Letting $x = \pi/2$ in this formula:

$$\frac{1}{\pi/2} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right) \implies \frac{\pi}{2} = \prod_{n=1}^{\infty} \left(\frac{4n^2}{4n^2 - 1}\right)$$

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John Wallis discovered this product while examining $\int_0^{\pi}\sin^n x dx$ for even and odd values of n. In 2015 researchers C. R. Hagen and Tamar Friedmann, in a surprise discovery, found the same formula in quantum mechanical calculations of the energy levels of a hydrogen atom.

$\zeta(4)$

Recall $\frac{\sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} \mp \cdots = \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right)$. Comparing x^4,

$$\sum_{\substack{m,n=1 \\ m<n}}^{\infty} \frac{1}{m^2 n^2 \pi^4} = \frac{1}{120}. \qquad (\star)$$

Using the multinomial expansion for squares,

$$\sum_{\substack{m,n=1 \\ m<n}}^{\infty} \frac{1}{m^2 n^2} = \frac{\sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{m=1}^{\infty} \frac{1}{m^4}}{2}.$$

Substituting $\zeta(2) = \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6}$ and multiplying $(\star)$ by π^4,

$$\frac{\pi^4}{120} = \frac{\pi^4/36 - \zeta(4)}{2} \implies \zeta(4) = \frac{\pi^4}{90}.$$

Even Riemann Zeta Values

Definition. The **Bernoulli numbers** are given by the recurrence relation:

- $B_0 = 1$,
- $B_0 + 2B_1 = 0$,
- $B_0 + 3B_1 + 3B_2 = 0$,
- $B_0 + 4B_1 + 6B_2 + 4B_3 = 0$.

In general for $k \geq 1$, $\sum_{j=0}^{k} \binom{k+1}{j} B_j = 0$. We can then construct a table:

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$\frac{-1}{2}$	$\frac{1}{6}$	0	$\frac{-1}{30}$	0	$\frac{1}{42}$	0	$\frac{-1}{30}$	0	$\frac{5}{66}$

For any positive even integer $2n$, we can compute the Riemann zeta function:

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}.$$

Bernoulli Numbers

Jacob Bernoulli determined a formula for the finite sum of powers of consecutive integers:

$$S_k(n) = 1^k + 2^k + \cdots + (n-1)^k.$$

If k is small, then it is relatively easy to find formulas for $S_k(n)$, namely:

- $S_1(n) = \frac{1}{2}n^2 - \frac{1}{2}n.$
- $S_2(n) = \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n.$
- $S_3(n) = \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2.$
- $S_4(n) = \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n.$

Adjusting the coefficients of the polyomial $S_k(n)$,

$$S_k(n) = \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j n^{k+1-j}.$$