Fractals
Lecture 12

Justin Stevens
Outline

1. Fractals
   - Koch Snowflake
   - Hausdorff Dimension
   - Sierpinski Triangle
   - Mandelbrot Set
Definition. The Koch snowflake can be constructed by starting with an equilateral triangle, then recursively altering each line segment as follows:
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**Definition.** The Koch snowflake can be constructed by starting with an equilateral triangle, then recursively altering each line segment as follows:

- Divide the line segment into three segments of equal length.
- Draw an equilateral triangle that has the middle segment from the previous step as its base and points outward.
- Remove the line segment that is the base of the new triangle.
Figure 1: The Koch snowflake is one of the earliest discovered fractal curves. It is based on a 1904 paper titled “On a continuous curve without tangents, constructible from elementary geometry" by the Swedish mathematician Helge von Koch.
Perimeter of the Koch Snowflake

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As the number of iterations tends to infinity, the limit of the perimeter is:

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Area of the Koch Snowflake

In each iteration a new triangle is added on each side of the previous iteration, so the number of new triangles added in iteration \( j \) is

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As \( n \to \infty \), this is an infinite geometric series with first term 1/3:

\[
A_n = a_0 \left(1 + \frac{1/3}{1 - 4/9}\right) = 8/5 \cdot a_0.
\]
Hausdorff Dimension

**Definition.** The **Hausdorff dimension** measures the *roughness* of a metric space. If $S$ is the scaling factor and $N$ is the mass-scaling factor, then

$$N = S^D \iff D = \log_S(N) = \frac{\log N}{\log S}.$$
Hausdorff Dimension of Koch Snowflake

\[ 3^D = 4 \]
\[ D = \log_3(4) \approx 1.262 \]

Scaling factor: \( \frac{1}{3} \)

Mass scaling factor: \( \frac{1}{4} \)

Figure 2: 3Blue1Brown
**Definition.** The Sierpinski triangle starts with an equilateral triangle and recursively divides each triangle into four smaller congruent equilateral triangles and removes the central one.
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The Hausdorff Dimension is given by $2^D = 3 \iff D = \log_2 3 \approx 1.585$. 
Figure 3: Take Pascal’s triangle with $2^n$ rows and color the even numbers white and the odd numbers black
“How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension" is a 1967 paper by Benoit Mandelbrot.

Figure 4: If the coastline of Great Britain is measured using units 100 km long, then the length of the coastline is approximately 2,800 km. With 50 km units, the total length is approximately 3,400 km, approximately 600 km longer.
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\[ P_c : z \mapsto z^2 + c, \]

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$$(0, P_c(0), P_c(P_c(0)), P_c(P_c(P_c(0))), \cdots)$$

obtained by iterating $P_c(z)$ starting at critical point $z = 0$. This either escapes to infinity or stays within a disk of some finite radius. The Mandelbrot set is defined as the set of all points $c$ such that the above sequence does not escape to infinity. Specifically, $|P_n^c(0)| \leq 2$ for all $n \geq 0$. 

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First Mandelbrot Set

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Modern Mandelbrot Set