Number Theory with python PROGRAMMING

Recursion

Lecture 9

Justin Stevens

Outline

Induction

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- **Base Case:** Show $P(1)$.
- **Inductive Step:** Show $P(k)$ implies $P(k+1)$ for any integer $k \geq 1$.

The assumption we make is known as the induction hypothesis.

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Adding the next cube to both sides of our assumption gives:

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This is the sum of cubes formula for $n = k + 1$, hence the identity holds.

Proof Without Words of Sum of Cubes

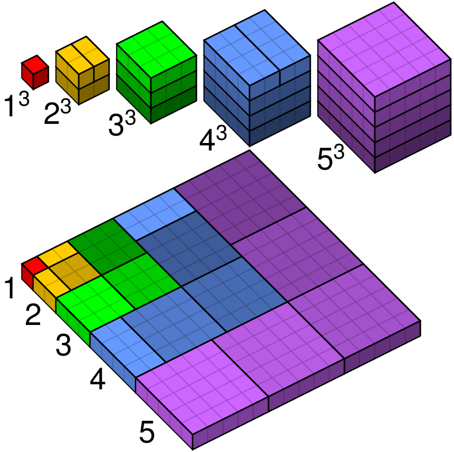

Figure 1: Alan L. Fry, Mathematical Association of America (1985)

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From the reduced residue system, there exists i with $i \cdot 2^k + A \equiv 0 \pmod{5}$. Then N_i is a $k+1$-digit number divisible by 5^{k+1} all of whose digits are odd.

Strong Induction

Theorem. (Strong Induction) Suppose we have a statement P we wish to show is true for all positive integers. We can prove this in two steps:

- Base Case: Show that the statement is true for $P(1)$.
- Inductive Step: If k is an arbitrary integer ≥ 1 such that $P(1), P(2), \cdots, P(k)$ are true, show $P(k+1)$ is also true.

USAMO Good Numbers

Example 1. (USAMO) We call integer n *good* if we can write $n = a_1 + a_2 + \cdots + a_k$, where $a_1, a_2, \cdots, a_k$ are not necessarily distinct positive integers satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} = 1.$$

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Observe that if n is good, then

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$$\frac{1}{2a_1} + \cdots + \frac{1}{2a_k} + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1 \implies 2n + 9 \text{ is good.}$$

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Let $P(n)$ be the proposition "$n, n+1, n+2, \cdots, 2n+7$ are good".

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The base case $P(33)$ is given. If k is good, then $2k+8$ and $2k+9$ are also, so $P(k) \implies P(k+1)$. By induction, every integer ≥ 33 is good. □

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$$5 = 2 + 3, \ \ 7 = 3 + 4, \ \ 11 = 2 + 9, \ \ 13 = 4 + 9, \ \ 17 = 8 + 9.$$

Multiplying the terms of these represenations by 2:

$$10 = 4 + 6, \ \ 14 = 6 + 8, \ \ 22 = 4 + 18, \ \ 26 = 8 + 18, \ \ 34 = 16 + 18.$$

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To find a representation for an odd number, such as 2001, we find the largest power of 3 less than the number: $3^6 = 729 < 2001 < 2187 = 3^7$.

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Recall $2001 = 729 + 1272$. By the hypothesis, 1272 can be written in the given form. We now show no term of this sum divides 729 and vice versa.

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Since $1272 = 2 \cdot 636$, multiplying the representation of 636 by 2 gives one for 1272. Also since $636 < 729$, no term of this sum will be divisible by 729.

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Factoring out 2, $2001 = 729 + 8 \cdot 159$. Finding the largest power of 3,

$$159 = 81 + 78 = 81 + 2 \cdot 39 = 81 + 2 \cdot (27 + 12) = 81 + 54 + 24.$$

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Substituting this into our equation above, our desired representation is

$$2001 = 729 + 8 \cdot 159 = 729 + 8 \cdot (81 + 54 + 24) = \boxed{729 + 648 + 432 + 192}.$$

Putnam Representation Problem Rigorous Proof

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$$n' = \frac{n - 3^s}{2} < \frac{3^{s+1} - 3^s}{2} = 3^s.$$

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  $$n' = \frac{n - 3^s}{2} < \frac{3^{s+1} - 3^s}{2} = 3^s.$$

  Hence none of the terms of the representation for $2n'$ are divisible by 3^s. Also since they are all even, none divide 3^s. Putting together the representations for $2n'$ with 3^s gives a valid representation for n. □

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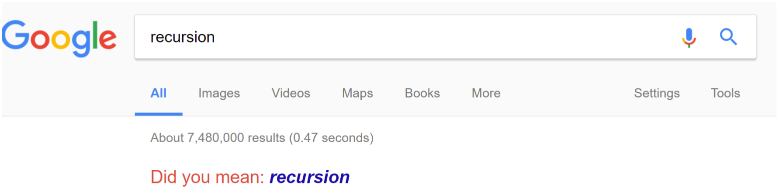

McCarthy 91 Function

Example. The McCarthy 91 function is a recursive function defined by

$$M(n) = \begin{cases} n - 10, & \text{if } n > 100 \\ M(M(n+11)), & \text{if } n \leq 100. \end{cases}$$

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Proof. For a base case, if $90 \leq k < 101$, then we see $k + 11 > 100$, so

$$M(k) = M(M(k+11)) = M(k + 11 - 10) = M(k+1).$$

Therefore, $M(90) = M(91) = \cdots = M(100) = M(101) = 101 - 10 = 91.$

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We now use induction on blocks of 11 numbers. Assume that $M(k) = 91$ for $a \leq k < a + 11$. Then, for $a - 11 \leq k < a$,

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Since we established the base case $a = 90$, $M(k) = 91$ for any k in such a block. Letting a be multiples of 10, there are no holes between the blocks.

Ackermann Function

Example 2. The Ackermann function is a recursive function defined by

$$A(m,n) = \begin{cases} n+1, & \text{if } m = 0 \\ A(m-1,1), & \text{if } n = 0 \\ A(m-1, A(m, n-1)) & \text{otherwise.} \end{cases}$$

Prove that for every integer $n \geq 0$,

$$A(1,n) = n+2, \quad A(2,n) = 2n+3, \quad A(3,n) = 2^{n+3} - 3.$$

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   Therefore, $A(1,n) = n+2$ for all $n \geq 0$ by induction.

2. When $n = 0$, $A(2,0) = A(1,1) = 3$. Assume $A(2,k) = 2k+3$, so

   $$A(2,k+1) = A(1, A(2,k)) = A(1, 2k+3) = (2k+3)+2 = 2(k+1)+3.$$

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Ackermann Function Proof for $m = 3$

Proof. Recall the definition of the Ackermann function,

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Computerphile Video

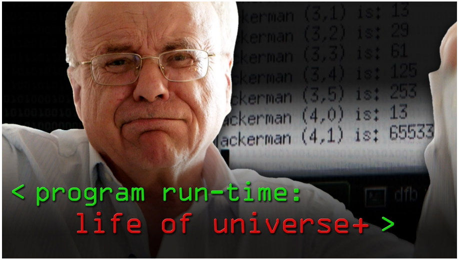

Figure 2: In computability theory, the Ackermann function is the earliest-discovered example of a total computable function that is not primitive recursive.

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- In general, we define the up-arrow notation recursively as

$$a \uparrow^n b = \begin{cases} 1 & \text{if } n \geq 1 \text{ and } b = 0 \\ a \uparrow^{n-1} (a \uparrow^n (b-1)) & \text{otherwise.} \end{cases}$$

Ackermann Function For $m \geq 4$

We can compute $A(4,0) = 13$ and $A(4,1) = 65533$. For $n = 2$,

$$A(4,2) = 2^{2^{2^{2^2}}} - 3 = 2^{65536} - 3.$$

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Since the Ackermann functions grows very fast and utilizes deep recursion, it can be used as a benchmark of a compiler's ability to optimize recursion.

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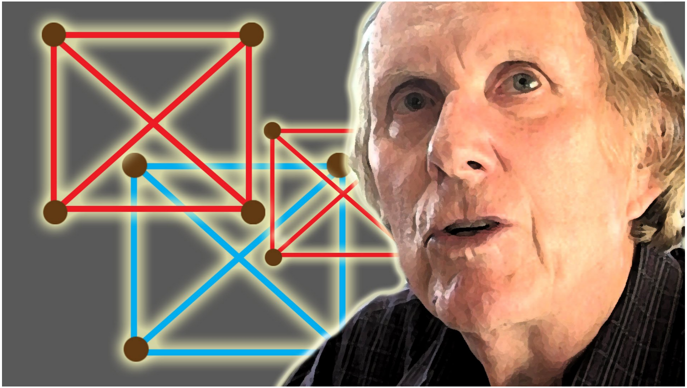
Graham's Number Tower

Finally, Graham's number is a stacked up-arrow tower:

$$G = \left.\begin{matrix} 3\underbrace{\uparrow\uparrow\cdots\cdots\cdots\cdots\cdots\cdots\uparrow}3 \\ 3\underbrace{\uparrow\uparrow\cdots\cdots\cdots\cdots\uparrow}3 \\ \vdots \\ \underbrace{\qquad\qquad\qquad} \\ 3\underbrace{\uparrow\uparrow\cdots\cdots\uparrow}3 \\ 3\uparrow\uparrow\uparrow\uparrow 3 \end{matrix}\right\}\text{64 layers}$$

Ronald Graham on Numberphile



Outline

Relevant Links

- Alan L. Fry: Proof without words (Sum of cubes)
- Computerphile: The Most Difficult Program to Compute?
- Wait But Why: From 1,000,000 to Graham's Number
- Numberphile: What is Graham's Number (feat Ron Graham)
- Numberphile: The mystery of 0.577
- Albert R. Meyer: Book Stacking Video (MIT 6.042J)
- Brian Brushwood: The Leaning Tower of Cardsa