Mock AMC 12

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May 21, 2012

Rules

You have 75 minutes to complete this test. You may only use graph paper, scrap paper, a protractor, a compass, a ruler, a pencil, and an eraser. All questions are multiple choice and have only one answer. Please submit your answers in the following google form: (will make later). You get 6 points for a correct answer, 1.5 for a blank submission, and 0 points for an incorrect answer.

Qualifying for Mock USAMO

This test goes along with the 2012 Mock AIME II. If you participated in the Mock AIME II, your index for qualification for the Mock USAMO will be (your mock AMC 12 score) + $10\times$(your mock AIME score). If you did not participate in the mock AIME II, your index for qualification for Mock USAMO will be mock AMC 12 score+10. The top 50% of all participants will get to participate in the 2012 Mock USAMO.

Good luck!

1. Determine the value of $\dfrac{2012^2 - 1983^2}{2012 + 1983}$.

(A) 29 **(B)** 39 **(C)** 1006 **(D)** 3995 **(E)** 4005

2. A set of 31 real numbers has a sum of 104. A new set is formed by taking each number in the old set and increasing it by 3. For instance, a 1 in the old set would become a 4 in the new set. What is the sum of the numbers in the new set?

(A) 31 **(B)** 104 **(C)** 135 **(D)** 166 **(E)** 197

3. Let a, b, and c be positive real numbers that satisfy the system

$$\begin{aligned} a^2 + b^2 &= 2(8 - ab), \\ b^2 + c^2 &= 2(18 - bc), \\ a^2 + c^2 &= 2(32 - ac). \end{aligned}$$

What is $\min(a, b, c)$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

4. A list of 2011 real numbers arranged in a ring has a sum of 2011. We create a new list of 2011 real numbers by writing the average of each 2 consecutive terms (For example, if a list was 1, 2, 3, our new list would be 1.5, 2.5, 2, and the next iteration would be 2, 1.75, 2.25, etc.) What is the maximum value for the sum of the elements in the new list?

(A) 1005 **(B)** 1005.5 **(C)** 1006 **(D)** 2011 **(E)** 4022

5. Rectangle $ABCD$ has $AB = 5$ and $BC = 12$. CD is extended to a point E such that $\triangle ACE$ is isosceles with base EC. What is the area of triangle ADE?

(A) 15 **(B)** 30 **(C)** 60 **(D)** 75 **(E)** 90

6. It takes Paula 5 hours to incorrectly paint a house and 17 hours to paint it correctly, and she has to paint three houses. When Paula feels lazy, she paints all three houses incorrectly, when she feels half-hearted, she paints one house correctly and two incorrectly, and when she feels motivated, she paints two houses correctly and one incorrectly. If each of these moods have an equal chance of occurring, what is the expected value of hours Paula will work painting the three houses?

(A) 25 **(B)** 26 **(C)** 27 **(D)** 28 **(E)** 29

7. A sequence is defined recursively as $a_n = 3a_{n-1} - 2a_{n-2}$, where $a_0 = 1$ and $a_1 = 3$. Compute a_{10}.

(A) 100 **(B)** 257 **(C)** 1001 **(D)** 1023 **(E)** 2047

8. A regular circular clock is drawn. Lines are drawn in-between A_1 and A_2, A_2 and A_3, A_3 and A_4 and so forth until a line is drawn in-between A_{11} and A_{12} where A_i corresponds with the number i on a regular clock. This creates regular 12-gon $A_1A_2A_3A_4A_5A_6A_7A_8A_9A_{10}A_{11}A_{12}$. Find the probability that a randomly chosen point inside the circle is not inside the given regular 12-gon.

(A) $\frac{\pi\sqrt{3}-1}{4}$ **(B)** $\frac{\pi-\sqrt{3}}{\pi}$ **(C)** $\frac{\pi\sqrt{3}-1}{\pi}$ **(D)** $\frac{\pi-3}{\pi}$ **(E)** $\frac{\pi-3}{4}$

9. After misplacing her problem on the 2012 AMCs, Paula the Painter is forced to paint a rectangular wall whose dimensions are integer foot-lengths such that the length exceeds the width by a wide margin. Due to Paulas ignorance, she does not have enough paint to cover the entire wall. If she decides to paint everywhere except for a 1 foot stripe along the bottom of the wall, she will not have enough paint and will not be able to cover an additional 5 square feet. However, if she decides to paint everywhere with the exception of a 2 foot strip along the right side of the wall, she will have an excess of 7 square feet of paint. Determine the smallest possible value of the area of the wall.

(A) 154 **(B)** 162 **(C)** 176 **(D)** 182 **(E)** 194

10. The polynomial $x^3 - Ax + 15$ has three real roots. Two of these roots sum to 5. What is $|A|$?

(A) 5 **(B)** 15 **(C)** 22 **(D)** 25 **(E)** 30

11. Justin, David, and Siddharth are all called upon to paint a fence in place of Paula. When Siddharth and Justin work together, they take 3 hours to paint the fence. When David and Siddharth work together, they take 4 hours. When David and Justin work together, they take 5 hours. How many hours does it take all three of them to paint the fence, working at the same time?

(A) 12 **(B)** $\frac{120}{47}$ **(C)** $\frac{135}{47}$ **(D)** $\frac{210}{47}$ **(E)** $\frac{123}{47}$

12. Right triangle $\triangle ABC$ has $AB = 15$, $BC = 20$, and a right angle at B. Point D is placed on AC such that $\triangle DAB$ is isosceles with base AD. The circumcircle of $\triangle DAB$ intersects line CB at point $E \neq B$. What is BE?

(A) $\frac{15}{4}$ **(B)** $\frac{15}{2}$ **(C)** $\frac{45}{4}$ **(D)** $\frac{25}{2}$ **(E)** 15

13. Let z be a complex number such that $|z - 1| \leq 1$ and $|z - 2| = 1$. Find the largest possible value of $|z|^2$.

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

14. A grasshopper is situated on the origin of the Cartesian plane. If the grasshopper is at a given point (m, n) it jumps to the point $(m + 6, n + 9)$ or $(m + 9, n + 6)$. After a finite sequence of jumps, the grasshopper lands on some point (m_k, n_k) such that the sum of the roots of the polynomial $18x^2 + m_k x + n_k$ is -2. How many different paths could the grasshopper have taken to reach all such (m_k, n_k)?

(A) 12 **(B)** 18 **(C)** 19 **(D)** 27 **(E)** 55

15. Compute the minimum value of $\sqrt{a^2+9}+\sqrt{(b-a)^2+1}+\sqrt{(c-b)^2+4}+\sqrt{(10-c)^2+1}$, where a, b, c, and d are real numbers.

(A) 7 **(B)** $4 + \sqrt{109}$ **(C)** $\sqrt{136}$ **(D)** 12 **(E)** $\sqrt{149}$

16. Fyodor was throwing paint buckets of equal size in pigeon holes of varying size after a bad day at work. He has one red bucket, two yellow buckets, and three blue buckets. If there are three pigeon holes, the first of which can hold one bucket, the second can hold two buckets, and the third can hold three buckets, how many ways can Fyodor distribute the six buckets among the holes? Note that order of buckets in each hole matters (height order of different colored buckets), and that same color buckets are not distinguishable.

(A) 36 **(B)** 50 **(C)** 54 **(D)** 60 **(E)** 72

17. How many positive integers n exist such that the sum of the first $2n$ positive perfect squares is an integer multiple of the sum of the first n positive odd integers?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

18. The Yankees and the Red Sox are playing a 10-game series. At no point in the series did the Red Sox win more games than the Yankees did, but at the end of the 10-game series, both the Yankees and the Red Sox had won the same amount of games. Assuming there are no ties in baseball, how many such ways can this occur?

(A) 25 **(B)** 36 **(C)** 42 **(D)** 64 **(E)** 252

19. Let x_1, x_2, x_3, x_4 be a sequence of positive integers. Given that

$$\sum_{i=1}^{4} (3^{x_i} + 2^{x_i}) = 930,$$

determine $\sum_{i=1}^{4} x_i$.

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

20. Two chords are constructed in a circle with radius 4 such that their intersection point lies on the circle, and the angle formed by the two chords has measure 30°. One of the chords has length 5. Given that the largest possible value for the length of the other chord is $\dfrac{\sqrt{m}+\sqrt{n}}{p}$, where m, n, and p are positive integers, what is $m+n+p$?

(A) 112 **(B)** 116 **(C)** 268 **(D)** 783 **(E)** 1151

21. Let N be a positive integer whose prime factorization is of the form $p_1^2 p_2 p_3$, where p_1, p_2, and p_3 are all primes. If N is odd, and the sum of the positive divisors of N is 992, find the sum of all possible values of N?

(A) 525 **(B)** 634 **(C)** 782 **(D)** 1729 **(E)** 1886

22. Paula the painter has to paint four clients houses. The orders came in the form of color for the body of the house, and color for the roof, denoted as color-color. They each requested red-red, yellow-blue, red-blue, and yellow-yellow. Paula, being disorganized, misplaced her order sheets. If she randomly paints the houses bodies and roofs with her infinite paint buckets filled with either red, blue, or yellow, what is the probability she leaves one customer happy and the others unhappy? Note that for a customer to be happy, both parts of their houses must be painted correctly.

(A) $\dfrac{1378}{3^8}$ **(B)** $\dfrac{1538}{3^8}$ **(C)** $\dfrac{1788}{3^8}$ **(D)** $\dfrac{2048}{3^8}$ **(E)** $\dfrac{3070}{3^8}$

23. Let ABC be a right triangle with the right angle at B and $AB = 24$ and $BC = 7$. Let the incircle of $\triangle ABC$ have center D and touch AC at E. Let F be the intersection of the angle bisector of $\angle ACB$ with BA. Find the ratio of the area of $\triangle DBE$ to $\triangle DEF$.

(A) $\dfrac{7}{20}$ **(B)** $\dfrac{11}{20}$ **(C)** $\dfrac{15}{23}$ **(D)** $\dfrac{17}{25}$ **(E)** $\dfrac{37}{50}$

24. Let S_k, where $k \in \{1, 2, \ldots, 100\}$ denote the sum of the infinite geometric series whose first term is $\frac{k-1}{k!}$ and with common ratio $1/k$. Find the value of

$$\frac{100^2}{100!} + \sum_{k=1}^{100} \left|(k^2 - 3k + 1)S_k\right|.$$

(A) $\dfrac{100}{99!}$ **(B)** 2 **(C)** 3 **(D)** 4 **(E)** $\dfrac{100!}{99}$

25. Let Γ be a circle with center O and radius 4 with tangents from points A and C on Γ such that the intersection of the tangents, B, has $AB = 3$. Extend AO to meet Γ at E, extend CO to meet Γ at L with AL having midpoint K, and let EC intersect AB at D. If the midpoint of DC is labeled J, find the area of $BJCLK$.

(A) 9 **(B)** 10 **(C)** 12 **(D)** 15 **(E)** 18