Welcome to the 2013 Mock AIME!

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This is a test meant to represent the real AIME. Therefore, please treat this as a real AIME.

The only materials you may use are:

A proctractor, ruler, compass, pen, pencil, paper, and a computer for submitting answers. **NO** calculators, wolframalpha, asking others for help, etc.

To submit, a link is provided on the topic.

Good luck!

1. Two circles C_1 and C_2, each of unit radius, have centers A_1 and A_2 such that $A_1A_2 = 6$. Let P be the midpoint of A_1A_2 and let C_3 be a circle externally tangent to both C_1 and C_2. C_1 and C_3 have a common tangent that passes through P. If this tangent is also a common tangent to C_2 and C_1, find the radius of circle C_3. (tc1729)

2. Find the number of ordered positive integer pairs (a, b, c) such that a evenly divides b, $b+1$ evenly divides c, and $c - a = 10$. (Binomial-theorem)

3. Let $\lfloor x \rfloor$ be the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$. If $x = (7 + 4\sqrt{3})^{2^{2013}}$, compute $x\,(1 - \{x\})$. (tc1729)

4. Compute the number of ways to fill in the following magic square such that:

• the product of all rows, columns, and diagonals are equal(the sum condition is waived),

• all entries are **non-negative** integers less than or equal to ten, and

• entries CAN repeat in a column, row, or diagonal.

1	9	
3		

(ksun48)

5. In quadrilateral $ABCD$, $AC \cap BD = M$. Also, $MA = 6, MB = 8, MC = 4, MD = 3$, and $BC = 2CD$. The perimeter of $ABCD$ can be expressed in the form $\frac{p\sqrt{q}}{r}$ where p and r are relatively prime, and q is not divisible by the square of any prime number. Find $p + q + r$. (Binomial-theorem)

6. Find the number of integer values k can have such that the equation

$$7\cos x + 5\sin x = 2k + 1$$

has a solution. (tc1729)

7. Let S be the set of all 7th primitive roots of unity with imaginary part greater than 0. Let T be the set of all 9th primitive roots of unity with imaginary part greater than 0. (A primitive nth root of unity is a nth root of unity that is not a kth root of unity for any $1 \le k < n$.)Let $C = \sum_{s\in S}\sum_{t\in T}(s+t)$. The absolute value of the real part of C can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime numbers. Find $m + n$.(Binomial-theorem)

8. Let $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{v}$ be two perpendicular vectors in the $x - y$ plane. If there are n vectors $\mathbf{r}_i$ for $i = 1, 2, \ldots, n$ in the same plane having projections of 1 and 2 along $\mathbf{u}$ and $\mathbf{v}$ respectively, then find

$$\sum_{i=1}^{n} \|\mathbf{r}_i\|^2.$$

(Note: $\mathbf{i}$ and $\mathbf{j}$ are unit vectors such that $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$, and the projection of a vector $\mathbf{a}$ onto $\mathbf{b}$ is the length of the vector that is formed by the origin and the foot of the perpendicular of $\mathbf{a}$ onto $\mathbf{b}$.). (tc1729)

9. In a magic circuit, there are six lights in a series, and if one of the lights short circuit, then all lights after it will short circuit as well, without affecting the lights before it. Once a turn, a random light that isn't already short circuited is short circuited. If E is the expected number of turns it takes to short circuit all of the lights, find $100E$. (admin25)

10. Let T_n denote the nth triangular number, i.e. $T_n = 1 + 2 + 3 + \cdots + n$. Let m and n be relatively prime positive integers so that

$$\sum_{i=3}^{\infty}\sum_{k=1}^{\infty}\left(\frac{3}{T_i}\right)^k = \frac{m}{n}.$$

Find $m + n$. (djmathman)

11. Let a, b, and c be the roots of the equation $x^3 + 2x - 1 = 0$, and let X and Y be the two possible values of $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$. Find $(X + 1)(Y + 1)$. (djmathman)

12. In acute triangle ABC, the orthocenter H lies on the line connecting the midpoint of segment AB to the midpoint of segment BC. If $AC = 24$, and the altitude from B has length 14, find $AB \cdot BC$. (admin25)

13. In acute $\triangle ABC$, H is the orthocenter, G is the centroid, and M is the midpoint of BC. It is obvious that $AM \geq GM$, but $GM \geq HM$ does not always hold. If $[ABC] = 162$, $BC = 18$, then the value of GM which produces the smallest value of AB such that $GM \geq HM$ can be expressed in the form $a + b\sqrt{c}$, for b squarefree. Compute $a + b + c$. (ksun48)

14. Let $P(x) = x^{2013} + 4x^{2012} + 9x^{2011} + 16x^{2010} + \cdots + 4052169x + 4056196 = \sum_{j=1}^{2014} j^2 x^{2014-j}$. If $a_1, a_2, \cdots a_{2013}$ are its roots, then compute the remainder when $a_1^{997} + a_2^{997} + \cdots + a_{2013}^{997}$ is divided by 997. (ksun48)

15. Let S be the set of integers n such that $n|(a^{n+1} - a)$ for all integers a. Compute the remainder when the sum of the elements in S is divided by 1000. (ksun48)