A Collection of Problems

1. (NIMO Summer Problem 4) Let P be a function defined by $P(t) = a^t + b^t$, where a and b are complex numbers. If $P(1) = 7$ and $P(3) = 28$, compute $P(2)$.

2. (2012 Mock AMC 12 Problem 8) A regular circular clock is drawn. Lines are drawn in-between A_1 and A_2, A_2 and A_3, A_3 and A_4 and so forth until a line is drawn in-between A_{11} and A_{12} where A_i corresponds with the number i on a regular clock. This creates regular 12-gon $A_1A_2A_3 \cdots A_{12}$. Find the probability that a randomly chosen point inside the circle is not inside the given regular 12-gon.

   (A) $\frac{\pi\sqrt{3}-1}{4}$ **(B)** $\frac{\pi-\sqrt{3}}{\pi}$ **(C)** $\frac{\pi\sqrt{3}-1}{\pi}$ **(D)** $\frac{\pi-3}{\pi}$ **(E)** $\frac{\pi-3}{4}$

3. (2015 Mock AIME I Problem 4) At the AoPS Carnival, there is a "Weighted Dice" game show. This game features two identical looking weighted 6 sided dice. For each integer $1 \le i \le 6$, Die A has $\frac{i}{21}$ probability of rolling the number i, while Die B has a $\frac{7-i}{21}$ probability of rolling i. During one session, the host randomly chooses a die, rolls it twice, and announces that the sum of the numbers on the two rolls is 10. Let P be the probability that the die chosen was Die A. When P is written as a fraction in lowest terms, find the sum of the numerator and denominator.

4. (2012 Mock AMC 12 Problem 21) Let N be a positive integer whose prime factorization is of the form $p_1^2p_2p_3$, where p_1, p_2, and p_3 are all primes. If N is odd, and the sum of the positive divisors of N is 992, what is N?

   (A) 525 **(B)** 634 **(C)** 782 **(D)** 1729 **(E)** 1886

5. (2015 NIMO Summer Problem 12) Let ABC be a triangle whose angles measure A, B, C, respectively. Suppose $\tan A$, $\tan B$, $\tan C$ form a geometric sequence in that order. If $1 \le \tan A + \tan B + \tan C \le 2015$, find the number of possible integer values for $\tan B$. (The values of $\tan A$ and $\tan C$ need not be integers.)

6. (2013 Mock AIME I Problem 5) In quadrilateral $ABCD$, $AC \cap BD = M$. Let $MA = 6, MB = 8, MC = 4, MD = 3$, and $BC = 2CD$. The perimeter of $ABCD$ can be expressed in the form $\frac{p\sqrt{q}}{r}$ where p and r are relatively prime, and q is not divisible by the square of any prime number. Find $p + q + r$.

7. (2012 Mock AIME II Problem 14) Call a number a *near Carmichael number* if for all prime divisors p of the *near Carmichael number*, n, $(p-1)$ divides $(n-1)$ and n is not prime. Find the sum of all two digit *near Carmichael numbers*.

8. (NIMO 20 Problem 4) Let $f(n) = \frac{n}{3}$ if n is divisible by 3 and $f(n) = 4n - 10$ otherwise. Find the sum of all positive integers c such that $f^5(c) = 2$. (Here $f^5(x)$ means $f(f(f(f(f(x)))))$.)

9. (2012 Mock AIME II Problem 15) Define $a_n = \sum_{i=0}^{n} f(i)$ for $n \ge 0$ and $a_n = 0$. Given that $f(x)$ is a polynomial, and $a_1, a_2 + 1, a_3 + 8, a_4 + 27, a_5 + 64, a_6 + 125, \cdots$ is an arithmetic sequence, find the smallest positive integer value of x such that $f(x) < -2012$.

10. (NIMO 15 Problem 7) Find the sum of the prime factors of 67208001, given that 23 is one without the use of a calculator.

11. (2015 Mock AIME I Problem 6) Find the number of 5 digit numbers using only the digits 1,2,3,4,5,6,7,8 such that every pair of adjacent digits is no more than 1 apart. For instance, 12345 and 33234 are acceptable numbers, while 13333 and 56789 are not.

12. (2013 Mock AIME I Problem 7) Let S be the set of all 7th primitive roots of unity with imaginary part greater than 0. Let T be the set of all 9th primitive roots of unity with imaginary part greater than 0. (A primitive nth root of unity is a nth root of unity that is not a kth root of unity for any

$1 \le k < n$.)Let $C = \sum_{s \in S} \sum_{t \in T} (s+t)$. The absolute value of the real part of C can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime numbers. Find $m+n$.

13. (2015 Mock AIME I Problem 8) Let a, b, c be consecutive terms (in that order) in an arithmetic sequence with common difference d. Suppose $\cos b$ and $\cos d$ are roots of a monic quadratic $p(x)$ with $p(-\frac{1}{2}) = \frac{1}{2014}$. Then

$$|\cos a + \cos b + \cos c + \cos d| = \frac{p}{q}$$

for positive relatively prime integers p and q. Find the remainder when $p+q$ is divided by 1000.

14. (2015 Mock AIME I Problem 9) Compute the number of positive integer triplets (a, b, c) such that $1 \le a, b, c \le 500$ which satisfy the following properties:

(a) abc is a perfect square,

(b) $(a+7b)c$ is a power of 2,

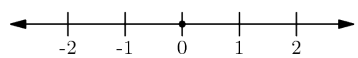
(c) a is a multiple of b.

15. (NIMO 15 Problem 5) Let r, s, t be the roots of the polynomial $x^3 + 2x^2 + x - 7$. Then

$$\left(1 + \frac{1}{(r+2)^2}\right)\left(1 + \frac{1}{(s+2)^2}\right)\left(1 + \frac{1}{(t+2)^2}\right) = \frac{m}{n}$$

for relatively prime positive integers m and n. Compute $100m + n$.

16. (2015 Mock AIME I Problem 12) Alpha and Beta play a game on the number line below.



Both players start at 0. Each turn, Alpha has an equal chance of moving 1 unit in either the positive or negative directions while Beta has a $\frac{2}{3}$ chance of moving 1 unit in the positive direction and a $\frac{1}{3}$ chance of moving 1 unit in the negative direction. The two alternate turns with Alpha going first. If a player reaches 2 at any point in the game, he wins; however, if a player reaches -2, he loses and the other player wins. If $\frac{p}{q}$ is the probability that Alpha beats Beta, where p and q are relatively prime positive integers, find $p+q$.

17. (NIMO 20 Problem 8) Justin the robot is on a mission to rescue abandoned treasure from a minefield. To do this, he must travel from the point $(0,0,0)$ to $(4,4,4)$ in three-dimensional space, only taking one-unit steps in the positive x, y, or z-directions. However, the evil David anticipated Justin's arrival, and so he has surreptitiously placed a mine at the point $(2,2,2)$. If at any point Justin is at most one unit away from this mine (in any direction), the mine detects his presence and explodes, thwarting Justin.

How many paths can Justin take to reach his destination safely?

18. (NIMO 19 Problem 5) Let a, b, c be complex numbers and p be a prime number. Assume that

$$a^n(b+c) + b^n(a+c) + c^n(a+b) \equiv 8 \pmod{p}$$

for each nonnegative integer n. Let m be the remainder when $a^p + b^p + c^p$ is divided by p, and k the remainder when m^p is divided by p^4. Find the maximum possible value of k.