ARML: Telescoping Series

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1 Lecture

With certain sums/products, the majority of the terms will cancel which helps to simplify calculations. Notation used throughout the document:

$$\prod_{i=1}^{n} a_i = a_1 \times a_2 \times a_3 \times \cdots \times a_n$$

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

Example 1.1 (Mathcounts)**.** *Evaluate the product*

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{6}\right)\left(1+\frac{1}{7}\right)$$

Solution. The product is equivalent to

$$\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\left(\frac{6}{5}\right)\left(\frac{7}{6}\right)\left(\frac{8}{7}\right) = \frac{8}{2} = 4$$

after cancellation □

Example 1.2. *Simplify the product*

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n}\right)$$

Solution.

$$\begin{aligned}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n}\right) &= \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{n-1}{n}\right)\\ &= \left(\frac{2}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right)\left(\frac{\cancel{4}}{\cancel{5}}\right)\cdots\left(\frac{\cancel{n-1}}{n}\right)\\ &= \frac{2}{n}\end{aligned}$$

□

Example 1.3. *Evaluate* $\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right)$

Solution.

$$\begin{aligned}\prod_{k=2}^{n}\left(1-\frac{1}{k^2}\right) &= \prod_{k=2}^{n}\frac{(k+1)(k-1)}{k^2}\\ &= \left(\prod_{k=2}^{n}\frac{k+1}{k}\right)\left(\prod_{k=2}^{n}\frac{k-1}{k}\right)\\ &= \left(\frac{n+1}{2}\right)\left(\frac{1}{n}\right)=\frac{n+1}{2n}\end{aligned}$$

□

Example 1.4 (AMC 12)**.** *Let* $T_n = 1+2+3+\cdots+n$ *and*

$$P_n = \frac{T_2}{T_2-1}\cdot\frac{T_3}{T_3-1}\cdot\frac{T_4}{T_4-1}\cdots\frac{T_n}{T_n-1}$$

for $n = 2,3,4,\cdots$. *Find* P_{1991}.

Solution. Notice that $T_n = \frac{n(n+1)}{2}$ and

$$T_n - 1 = \frac{n(n+1)}{2} - 1 = \frac{n^2+n-2}{2} = \frac{(n+2)(n-1)}{2}$$

. Therefore the product which we want to evaluate is equivalent to

$$\begin{aligned}P_n &= \left(\prod_{i=2}^{1991}\frac{i}{i+2}\right)\left(\prod_{i=2}^{1991}\frac{i+1}{i-1}\right)\\ &= \frac{2\times 3}{1992\times 1993}\times\left(\frac{1991\times 1992}{1\times 2}\right)\\ &= \frac{3\times 1991}{1993} = \frac{5973}{1993}\end{aligned}$$

□

Example 1.5. *Evaluate the sum*

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots+\frac{1}{99\times 100}$$

Solution. Notice that $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$, therefore the sum is equivalent to

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{99}-\frac{1}{100}\right)=\frac{1}{1}-\frac{1}{100}=\frac{99}{100}$$

□

2 Problem Solving

Here is a set of problems involving telescoping series. If you have any questions or want hints on any of these questions please feel free to ask me!

Problem 2.1 (AHSME)**.** Find the sum $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}+\cdots+\frac{1}{255\times 257}$

Problem 2.2. Find the product $\prod_{n=1}^{20}\left(1+\frac{2n+1}{n^2}\right)$.

Problem 2.3. Consider the sequence $1,-2,3,-4,5,-6,\cdots$ whose nth term is $(-1)^{n+1}\cdot n$. What is the average of the first 200 terms of the sequence?

Problem 2.4 (HMMT)**.** Evaluate $1\cdot 2-2\cdot 3+3\cdot 4-4\cdot 5+\cdots+2001\cdot 2002$.

Problem 2.5 (Mandelbrot)**.** Calculate

$$\prod_{n=1}^{13}\frac{n(n+2)}{(n+4)^2}$$

Problem 2.6 (AHSME)**.** Calculate

$$T=\frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2}.$$

Problem 2.7. Find

$$\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}}$$

Problem 2.8. Find the sum $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n-1}{n!}$

Problem 2.9 (AIME)**.** Let $x=\dfrac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}$. Find $(x+1)^{48}$.

Problem 2.10. Find the integer closest to $1000\displaystyle\sum_{n=3}^{1000}\frac{1}{n^2-4}$.

Problem 2.11. Evaluate $\displaystyle\sum_{k=2}^{n}k!(k^2+k+1)$

Problem 2.12 (Mandelbrot)**.** Compute the product

$$\frac{(1998^2-1996^2)\,(1998^2-1995^2)\cdots(1998^2-0^2)}{(1997^2-1996^2)\,(1997^2-1995^2)\cdots(1997^2-0^2)}$$

Problem 2.13 (USAMTS)**.** Determine the value of

$$S=\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{n^2}+\frac{1}{(n+1)^2}}+\cdots+\sqrt{1+\frac{1}{1999^2}+\frac{1}{2000^2}}$$

[Hint: This problem is very difficult. Try expressing each of the radicals in term of n]

Problem 2.14. Evaluate the infinite product $\displaystyle\prod_{n=2}^{\infty}\left(\frac{n^3-1}{n^3+1}\right)$ [Hint: Factor and write out the first few terms]