ARML: Polynomials

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Polynomials

1.1 Lecture

A polynomial $P(x)$ is defined as being of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_i x^i + a_1 x + a_0 \text{ for } 0 \leq i \leq n$$

We define the degree of a polynomial to be the highest degree of the polynomial. In the above form, $\deg P = n$.

Example 1.1.1. $P(x) = x^2 + 5x + 9$ *is a polynomial as is* $P(x) = \pi \times x^3 + \sqrt{3} \times x^2 + \frac{1}{9}$, *while* $P(x) = \frac{1}{x+2}$ *and* $P(x) = \sqrt{x}$ *are both* ***not*** *polynomials.*

Polynomials can similarly be defined in terms of their roots. Let r_1 be a root of $P(x)$ and we get

$$P(x) = (x - r_1)P_1(x) \text{ for } \deg P = n - 1$$

Now,

$$P_1(x) = (x - r_2)P_2(x) \text{ for } \deg P = n - 2$$

and repeating this process we arrive at

$$P(x) = a_n(x - r_1)(x - r_2)(x - r_3)\cdots(x - r_n)$$

Example 1.1.2 (Canada). *For real a, the polynomials $1988x^2 + ax + 8891 = 0$ and $8891x^2 + ax + 1988 = 0$ share a common root. Find all possible values of a.*

Solution. Subtract the two equations results in

$$(8891 - 1988)x^2 + (1998 - 8891) = 0 \implies x^2 - 1 = 0$$

Therefore, $x = \pm 1$. If $x = 1$, we get $a = -10879$, which results in both polynomials having the common root of $x = 1$. If $x = -1$, we get $a = 10879$, which results in both polynomials having the common root of $x = -1$.

The answer is therefore $a = \pm 10879$. □

Example 1.1.3 (ARML). *If $P(x)$ is a polynomial in x, and $x^{23}+23x^{17}-18x^{16} - 24x^{15} + 108x^{14} = (x^4 - 3x^2 - 2x + 9) \times P(x)$ for all values of x, compute the sum of the coefficients of $P(x)$.*

Solution. The sum of the coefficients of $P(x)$ is the same as $P(1)$. Now just plug $x = 1$ into the above equation. □

For a quadratic of the form $P(x) = a_2x^2 + a_1x + a_0$ and roots r_1, r_2, you are likely aware of the relations $r_1 + r_2 = \frac{-a_1}{a_2}$ and $r_1r_2 = \frac{a_0}{a_2}$. We will now attempt to derive similar relations for a cubic.

For $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, we can also express $P(x)$ in terms of its roots as $P(x) = a_3(x - r_1)(x - r_2)(x - r_3)$. Expanding this we get

$$\begin{aligned} P(x) = a_3(x - r_1)(x - r_2)(x - r_3) &= a_3\left(x^3 - x^2(r_1 + r_2 + r_3) + x(r_1r_2 + r_1r_3 + r_2r_3) - r_1r_2r_3\right) \\ &= a_3x^3 + a_2x^2 + a_1x + a_0 \end{aligned}$$

Setting the coefficients equal, we arrive at:

$$\begin{cases} r_1 + r_2 + r_3 = -\frac{a_2}{a_3} \\ r_1r_2 + r_1r_3 + r_2r_3 = \frac{a_1}{a_3} \\ r_1r_2r_3 = \frac{-a_0}{a_3} \end{cases}$$

Example 1.1.4. *Find the sum and the product of all the roots of* $x^3 + 2x^2 - 3x + 9 = 0$.

Solution. By the above relations, $r_1 + r_2 + r_3 = -\frac{2}{1}$ and $r_1 r_2 r_3 = -\frac{9}{1} = -9$. □

Example 1.1.5. *Suppose* $5x^3 + 4x^2 - 8x + 6 = 0$ *has three real roots* $a, b,$ *and* c. *Find the value of* $a(1 + b + c) + b(1 + a + c) + c(1 + a + b)$.

Example 1.1.6. *Find all ordered pairs* (x, y, z) *that satisfy*

$$\begin{aligned} x + y + z &= 17, \\ xy + xz + yz &= 94, \\ xyz &= 168 \end{aligned}$$

Solution. Set up the polynomial $f(a) = a^3 - 17a^2 + 94a - 168$ with roots x, y, z □

Problem 1.1.1 (A bit harder than the previous one)**.** Find all ordered pairs (x, y, z) that satisfy

$$\begin{aligned} x + y - z &= 0 \\ zx - xy + yz &= 27 \\ xyz &= 54 \end{aligned}$$

Before we delve into Vieta's formula, we need some notation.

$$\sigma_k = \sum_{1 \le a_1 \le a_2 \le \cdots \le a_k \le n} r_{a_1} r_{a_2} \cdots r_{a_k}$$

This notation may look incredibly intimidating, but I promise it is a lot easier than it looks. All it really is saying is that you are summing the product of k different numbers.

Example 1.1.7. *For $n = 3$, all possible products of 2 numbers are $\sigma_2 = r_1r_2 + r_1r_3 + r_2r_3$. All possible products of 1 number is $\sigma_1 = r_1 + r_2 + r_3$ and all possible products of 3 numbers are $\sigma_3 = r_1r_2r_3$. Therefore, we can rewrite the above cubic relations as*

$$\sigma_3 = \frac{-a_0}{a_3}, \sigma_2 = \frac{-a_2}{a_3}, \sigma_1 = \frac{a_1}{a_3}$$

Theorem 1.1.1 (Vieta's). *For a polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ and roots $r_1, r_2, \cdots, r_n$, we have*

$$\sigma_k = (-1)^k \times \frac{a_{n-k}}{a_n}$$

For example, for $n = 4$, we have

$$\begin{aligned}
\sigma_4 &= r_1r_2r_3r_4 = \frac{a_0}{a_4} \\
\sigma_3 &= r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 = \frac{-a_1}{a_4} \\
\sigma_2 &= r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = \frac{a_2}{a_4} \\
\sigma_1 &= r_1 + r_2 + r_3 + r_4 = \frac{-a_3}{a_4}
\end{aligned}$$

Example 1.1.8. *Find the product of the roots of $50x^{50} + 49x^{49} + \cdots + 1 = 0$.*

Solution. The product is $\sigma_{50} = (-1)^{50} \frac{a_0}{a_{50}} = \boxed{\frac{1}{50}}$. □

Example 1.1.9. *Three of the roots of $x^4 + ax^2 + bx + c = 0$ are $2, -3, 5$. Find the values of $a + b + c$.*

Solution. By Vieta's, $\sigma_1 = r_1 + r_2 + r_3 + r_4 = \frac{-a_3}{a_4}$, but $a_3 = 0$ since there is no x^3 term, therefore $r_1 + r_2 + r_3 + r_4 = 0$. We are given the roots $2, -3, 5$, therefore the fourth root must be -4. This gives

$$x^4 + ax^2 + bx + c = (x-2)(x-(-3))(x-(-4))(x-5)$$

Now set $x = 1$ in the above equation. □

Example 1.1.10. *Let $p(x) = x^3 - 5x^2 + 12x - 19$ have roots a, b, and c. Find the value of $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac}$.*

Solution. $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ac} = \frac{5}{19}$ □

Example 1.1.11. *Find $(2+r)(2+s)(2+t)(2+u)$ if r, s, t, and u are the roots of $f(x) = 3x^4 - x^3 + 2x^2 + 7x + 2$.*

Solution. $f(x) = 3(x-r)(x-s)(x-t)(x-u)$ The desired value is $\frac{f(-2)}{3} = \frac{2}{3}$ □

Example 1.1.12. *Find the two values of k for which $2x^3 - 9x^2 + 12x - k$ has a double root.*

Solution. Set the roots to be a, a, and b. Then $2a + b = \frac{9}{2}$ and $a^2 + 2ab = \frac{12}{2} = 6$. Solving this system gives $a = 1, 2$ which results in $k = 4, 5$. Checking gives $2x^3 - 9x^2 + 12x - 5 = (x-1)^2(2x-5)$ and $2x^3 - 9x^2 + 12x - 4 = (x-2)^2(2x-1)$. □

Example 1.1.13. *Suppose the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b, and c and the roots of $x^3 + rx^2 + sx + t = 0$ are $a+b, b+c, a+c$. (i) Find r. (ii) Find t.*

Solution. (i) Using Vieta's,

$$r = -\left[(a+b)+(b+c)+(a+c)\right] = -2\left(a+b+c\right)$$

Also, $a+b+c=-3$, therefore $r=-2\left(-3\right)=6$.
(ii) Using Vieta's,

$$t = -\left(a+b\right)\left(b+c\right)\left(a+c\right)$$

Now, also by Vieta's, we have $a+b+c=-3$. Therefore

$$t = -(-3-c)(-3-a)(-3-b)$$

Next, factor $x^3+3x^2+4x-11=(x-a)(x-b)(x-c)$. Substituting $x=-3$ into the above equation results in $\left(-3\right)^3+3\left(-3\right)^2+4\left(-3\right)-11=(-3-a)(-3-b)(-3-c)$ or therefore $(-3-a)(-3-b)(-3-c)=-23$. To finish, we get $t=-(3-c)(3-a)(3-b)=23$.

□

Example 1.1.14. *Let r_1, r_2, r_3 be the roots of the polynomial $5x^3-11x^2+7x+3$. Evaluate $r_1^3+r_2^3+r_3^3$.*

Solution. Use

$$r_1^3+r_2^3+r_3^3-3r_1r_2r_3 = (r_1+r_2+r_3)\left[r_1^2+r_2^2+r_3^2-(r_1r_2+r_1r_3+r_2r_3)\right]$$

□

Example 1.1.15 (Transformations)**.** *Given the polynomial $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ with roots $r_1, r_2, \cdots, r_n$. (i) Find the polynomial with roots $\frac{1}{r_1}, \frac{1}{r_2}, \cdots, \frac{1}{r_n}$. (ii) Find the polynomial with roots $mr_1, mr_2, \cdots, mr_n$. (iii) Find the polynomial with roots $r_1+1, r_2+1, \cdots, r_n+1$.*

Example 1.1.16. *Consider the polynomial $f(x)=x^3-9x^2+8x-17=0$ with roots a, b, c. Compute $\frac{1}{a+3}+\frac{1}{b+3}+\frac{1}{c+3}$.*

1.2 Problem Solving Time!

Problem 1.2.1. Let r_1, r_2, and r_3 be the three roots of the cubic x^3+3x^2+4x-4. Find the value of $r_1r_2+r_1r_3+r_2r_3$.

Problem 1.2.2. Suppose the polynomial $5x^3+4x^2-8x+6$ has three real roots a, b, and c. Find the value of $a(1+b+c)+b(1+a+c)+c(1+a+b)$.

Problem 1.2.3. Let m and n be the roots of the quadratic equation $4x^2+5x+3=0$. Find $(m+7)(n+7)$.

Problem 1.2.4. What is the sum of the reciprocals of the roots of the equation $\frac{2003}{2004}x+1+\frac{1}{x}=0$?

Problem 1.2.5. The equation $x^3-4x^2+5x-1.9=0$ has real roots r, s, and t. Find the length of an interior diagonal of a box with sides r, s, and t.

Problem 1.2.6. Determine $(r+s)(s+t)(t+r)$ if r, s, and t are the three real roots of the polynomial x^3+9x^2-9x-8.

Problem 1.2.7. Determine all real numbers a such that the two polynomials x^2+ax+1 and x^2+x+a have at least one root in common.

Problem 1.2.8. Let p, q, and r be the distinct roots of $x^3-x^2+x-2=0$. Find $p^3+q^3+r^3$.

Problem 1.2.9. Find the sum of the roots of $x^{2001}+\left(\frac{1}{2}-x\right)^{2001}=0$.

Problem 1.2.10. Let $P(x)$ be a quadratic polynomial with real coefficients satisfying $x^2-2x+2 \le P(x) \le 2x^2-4x+3$ for all real numbers x, and suppose $P(11)=181$. Find $P(16)$.

Problem 1.2.11. For certain real values of a, b, c, and d, the equation $x^4+ax^3+bx^2+cx+d=0$ has four nonreal roots. The product of two of these roots is $13+i$ and the sum of the other two roots is $3+4i$. Find b.

Problem 1.2.12. ζ_1, ζ_2, and ζ_3 are complex numbers such that

$$\zeta_1+\zeta_2+\zeta_3=1$$
$$\zeta_1^2+\zeta_2^2+\zeta_3^2=3$$
$$\zeta_1^3+\zeta_2^3+\zeta_3^3=7$$

Compute $\zeta_1^7+\zeta_2^7+\zeta_3^7$.

Problem 1.2.13. Let P be the product of the nonreal roots of $x^4 - 4x^3 + 6x^2 - 4x = 2005$. Find $\lfloor P \rfloor$.

Problem 1.2.14. The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Given that their sum is $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

Problem 1.2.15. For how many real numbers a does the quadratic equation $x^2 + ax + 6a = 0$ have only integer roots for x?

Problem 1.2.16. In the polynomial $x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$, the product of 2 of its roots is -32. Find k.

Problem 1.2.17. Consider the polynomials $P(x) = x^6 - x^5 - x^3 - x^2 - x$ and $Q(x) = x^4 - x^3 - x^2 - 1$. Given that z_1, z_2, z_3, and z_4 are the roots of $Q(x) = 0$, find $P(z_1) + P(z_2) + P(z_3) + P(z_4)$.

Problem 1.2.18. If α, β, γ are the roots of $x^3 - x - 1 = 0$, compute $\frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+\gamma}{1-\gamma}$

Problem 1.2.19. Let α_1 and α_2 be the roots of the quadratic $x^2 - 5x - 2 = 0$, and let β_1, β_2, and β_3 be the roots of the cubic $x^3 - 3x - 1 = 0$. Compute $(\alpha_1 + \beta_1)(\alpha_1 + \beta_2)(\alpha_1 + \beta_3)(\alpha_2 + \beta_1)(\alpha_2 + \beta_2)(\alpha_2 + \beta_3)$.