# Mock AIME I 2015

David Altizio, Justin Stevens

August 19, 2014

**Abstract**

This Mock AIME is the result of our efforts to produce a mock Mandelbrot contest through the summer. It is meant to be treated as a real mock AIME - in other words, three hours and no calculators. All the problems were written by us, and most of them have been kept secret until now. Good luck!

1. David, Justin, Richard, and Palmer are demonstrating a "math magic" concept in front of an audience. There are four boxes, labeled A, B, C, and D, and each one contains a different number. First, David pulls out the numbers in boxes A and B and reports that their product is 14. Justin then claims that the product of the numbers in boxes B and C is 16, and Richard states the product of the numbers in boxes C and D to be 18. Finally, Palmer announces the product of the numbers in boxes D and A. If $k$ is the number that Palmer says, what is $20k$?

2. Suppose that $x$ and $y$ are real numbers such that $\log_x 3y = \frac{20}{13}$ and $\log_{3x} y = \frac{2}{3}$. The value of $\log_{3x} 3y$ can be expressed in the form $\frac{a}{b}$ where $a$ and $b$ are positive relatively prime integers. Find $a+b$.

3. Let $A, B, C$ be points in the plane such that $AB = 25$, $AC = 29$, and $\angle BAC < 90^\circ$. Semicircles with diameters $\overline{AB}$ and $\overline{AC}$ intersect at a point $P$ with $AP = 20$. Find the length of line segment $\overline{BC}$.

4. At the AoPS Carnival, there is a "Weighted Dice" game show. This game features two identical looking weighted 6 sided dice. For each integer $1 \le i \le 6$, Die A has $\frac{i}{21}$ probability of rolling the number $i$, while Die B has a $\frac{7-i}{21}$ probability of rolling $i$. During one session, the host randomly chooses a die, rolls it twice, and announces that the sum of the numbers on the two rolls is 10. Let $P$ be the probability that the die chosen was Die A. When $P$ is written as a fraction in lowest terms, find the sum of the numerator and denominator.

5. In an urn there are a certain number (at least two) of black marbles and a certain number of white marbles. Steven blindfolds himself and chooses two marbles from the urn at random. Suppose the probability that the two marbles are of opposite color is $\frac{1}{2}$. Let $k_1 < k_2 < \cdots < k_{100}$ be the 100 smallest possible values for the total number of marbles in the urn. Compute the remainder when
$$k_1 + k_2 + k_3 + \cdots + k_{100}$$
is divided by 1000.

6. Find the number of 5 digit numbers using only the digits $1, 2, 3, 4, 5, 6, 7, 8$ such that every pair of adjacent digits is no more than 1 apart. For instance, 12345 and 33234 are acceptable numbers, while 13333 and 56789 are not.

7. For all points $P$ in the coordinate plane, let $P'$ denote the reflection of $P$ across the line $y = x$. For example, if $P = (3, 1)$, then $P' = (1, 3)$. Define a function $f$ such that for all points $P$, $f(P)$ denotes the area of the triangle with vertices $(0, 0)$, $P$, and $P'$. Determine the number of lattice points $Q$ in the first quadrant such that $f(Q) = 8!$.

8. Let $a, b, c$ be consecutive terms (in that order) in an arithmetic sequence with common difference $d$. Suppose $\cos b$ and $\cos d$ are roots of a monic quadratic $p(x)$ with $p(-\frac{1}{2}) = \frac{1}{2014}$. Then
$$|\cos a + \cos b + \cos c + \cos d\,| = \frac{p}{q}$$
for positive relatively prime integers $p$ and $q$. Find the remainder when $p + q$ is divided by 1000.

9. Compute the number of positive integer triplets $(a, b, c)$ with $1 \le a, b, c \le 500$ that satisfy the following properties:
   (a) $abc$ is a perfect square,
   (b) $(a + 7b)c$ is a power of 2,
   (c) $a$ is a multiple of $b$.

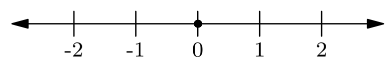
10. Let $f$ be a function defined along the rational numbers such that $f(\frac{m}{n}) = \frac{1}{n}$ for all relatively prime positive integers $m$ and $n$. The product of all rational numbers $0 < x < 1$ such that
$$f\left(\frac{x - f(x)}{1 - f(x)}\right) = f(x) + \frac{9}{52}$$
can be written in the form $\frac{p}{q}$ for positive relatively prime integers $p$ and $q$. Find $p + q$.

11. Suppose $\alpha$, $\beta$, and $\gamma$ are complex numbers that satisfy the system of equations

$$\alpha+\beta+\gamma=6,$$
$$\alpha^3+\beta^3+\gamma^3=87,$$
$$(\alpha+1)(\beta+1)(\gamma+1)=33.$$

If $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{m}{n}$ for positive relatively prime integers $m$ and $n$, find $m+n$.

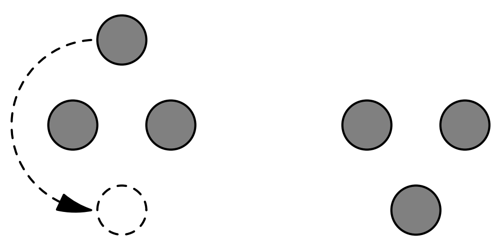
12. Alpha and Beta play a game on the number line below.




Both players start at 0. Each turn, Alpha has an equal chance of moving 1 unit in either the positive or negative directions while Beta has a $\frac{2}{3}$ chance of moving 1 unit in the positive direction and a $\frac{1}{3}$ chance of moving 1 unit in the negative direction. The two alternate turns with Alpha going first. If a player reaches 2 at any point in the game, he wins; however, if a player reaches $-2$, he loses and the other player wins. If $\frac{p}{q}$ is the probability that Alpha beats Beta, where $p$ and $q$ are relatively prime positive integers, find $p+q$.

13. Let $A_1A_2A_3A_4A_5A_6$ be a hexagon inscribed inside a circle of radius $r$. Furthermore, for each positive integer $1\le i\le 6$ let $M_i$ be the midpoint of the segment $\overline{A_iA_{i+1}}$, where $A_7\equiv A_1$. Suppose that hexagon $M_1M_2M_3M_4M_5M_6$ can also be inscribed inside a circle. If $A_1A_2=A_3A_4=5$ and $A_5A_6=23$, then $r^2$ can be written in the form $\frac{m}{n}$ where $m$ and $n$ are positive relatively prime integers. Find $m+n$.

14. Consider a set of $\frac{n(n+1)}{2}$ pennies laid out in the formation of an equilateral triangle with "side length" $n$. You wish to move some of the pennies so that the triangle is flipped upside down. For example, with $n=2$, you could take the top penny and move it to the bottom to accomplish this task, as shown:



Let $S_n$ be the minimum number of pennies for which this can be done in terms of $n$. Find $S_{50}$.

15. Let $\triangle ABC$ be a triangle with $AB=13$, $BC=14$, and $CA=15$. Let $O$ denote its circumcenter and $H$ its orthocenter. The circumcircle of $\triangle AOH$ intersects $AB$ and $AC$ at $D$ and $E$ respectively. Suppose $\frac{AD}{AE}=\frac{m}{n}$ where $m$ and $n$ are positive relatively prime integers. Find $m-n$.